10. Relativistische Elektrodynamik

10.1. Prinzipien der speziellen Relativitätstheorie (SRT)

Raum-Zeit-Koordinaten, Ereignisse

$x^\mu = (ct, \vec{x})$ Vierertupel

$x^0 = ct, \ x^1 = x, \ x^2 = y, \ x^3 = z$

Indizes 0, 1, 2, 3

Koordinaten abhängig vom Bezugssystem

ausgezeichnete Klasse: Inertialsysteme, K

[Teilchen bewegen sich gleichförmig ohne Kräfte]

nur lokal, eigentlich ART allgemeine Relativitätstheorie

da Gravitation

Postulat

(1) Naturgesetze identisch in allen Inertialsystemen K

$\Rightarrow$ Forminvarianz der Gleichungen

(2) Wechselwirkungen nicht instantan, endliche Ausbreitungsgeschwindigk.

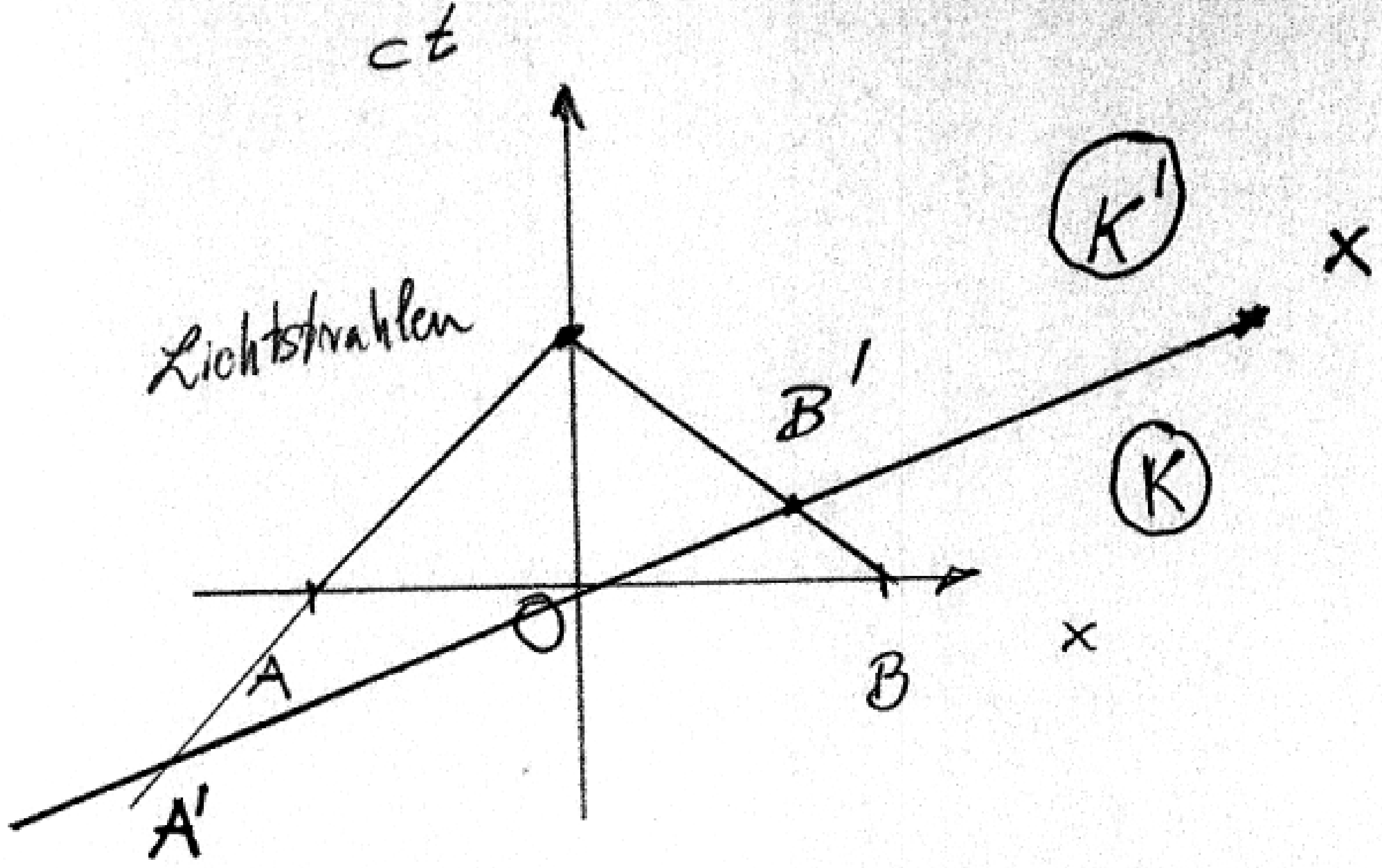
c, [Lichtgeschwindigkeit]

Folgerungen: (II)

(a) c identisch in allen K

(b) Gleichzeitigkeit nicht absolut

Minkowski-Diagramm



(c) Ereignisse P, Q mit Koordinaten x^μ, $x^\mu + dx^\mu$ im (K)

infinitesimaler Abstand ds

$$\boxed{ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2}$$

$ds = 0 \Rightarrow P, Q$ verbunden durch Lichtsignal

$\Rightarrow ds' = 0$ im (K')

allgemein Taylor

Homogenität in Raum und Zeit

$$ds^2 = a(V)\, ds'^2$$

Relativgeschwindigkeit (K) und (K')

Symmetrie

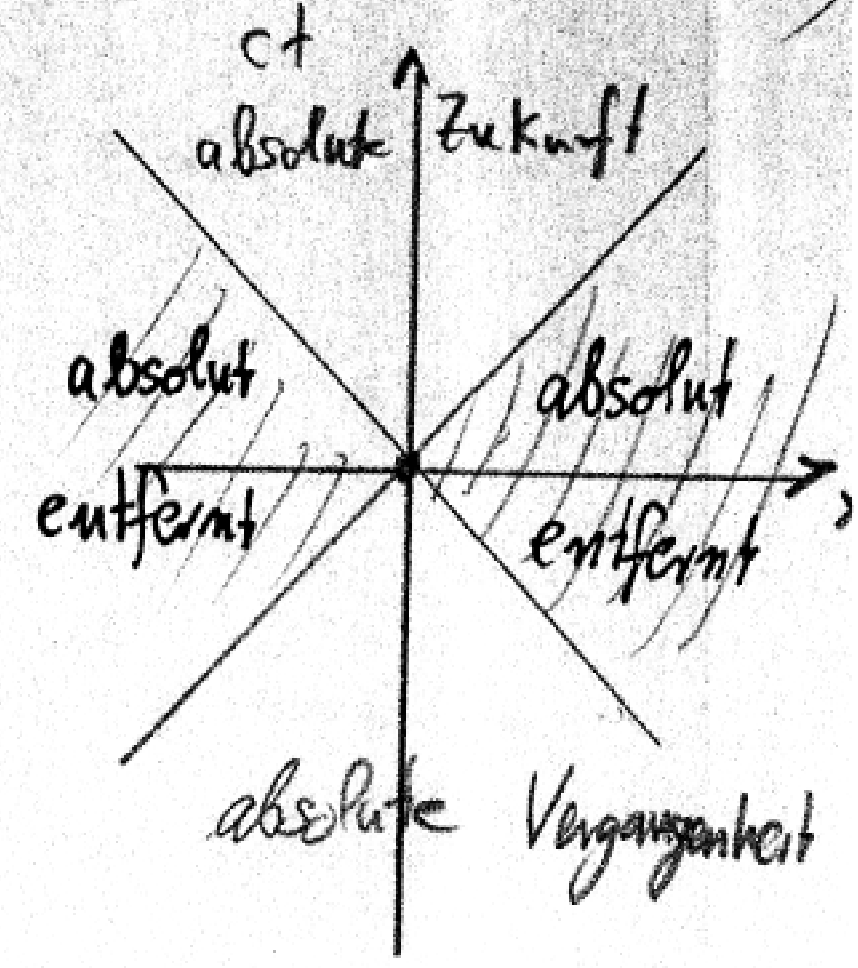
$$\boxed{ds^2 = ds'^2}$$ in allen Inertialsystemen!

Definition:

$$\boxed{\begin{array}{ll} ds^2 > 0 & \text{zeitartig} \\ ds^2 = 0 & \text{lichtartig} \\ ds^2 < 0 & \text{raumartig} \end{array}}$$

(nicht Kausal verbunden)

unabhängig vom Inertialsystem.



Bedeutung: Uhr bewegt sich in (K)

in dt um $|d\vec{x}|$

lokales Inertialsystem $d\vec{x}\,' = 0$

$$ds^2 = c^2 dt^2 - d\vec{x}^{\,2} =$$

$$= ds'^2 = c^2 dt'^2$$

$$\Rightarrow \boxed{dt' = \frac{ds}{c} = dt\sqrt{1 - v^2/c^2}}$$

Eigenzeit der bewegten Uhr

Warnung: Vorzeichenkonvention für ds^2

$(+\,-\,-\,-)$ Raumzeit

alternativ $(-\,+\,+\,+)$ [veraltet Minkowski $x^0 = ict$

$(+\,+\,+\,+)$]

Euklidisch mit imaginärer Zeit

Minkowski - Metrik

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Indexkonvention
oben : Kontravariant
unten : Kovariant

$$\boxed{ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu}$$

Einsteinsche Summenkonvention

Transformationen zwischen Inertialsystemen : Lorentz-Transformation

linearer Abbildung

$$x^\mu \longrightarrow \boxed{x'^\mu = \Lambda^\mu{}_\nu\, x^\nu} \qquad dx'^\mu = \Lambda^\mu{}_\nu\, dx^\nu$$

Invarianz $\quad \eta'_{\mu\nu} = \eta_{\mu\nu}$ aus Postulat

$$ds'^2 = \eta_{\mu\nu}(\Lambda^\mu{}_\sigma\, dx^\sigma)(\Lambda^\nu{}_\tau\, dx^\tau) \overset{!}{=} \eta_{\sigma\tau}\, dx^\sigma dx^\tau$$

$$\Rightarrow \boxed{\eta_{\mu\nu}\Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau = \eta_{\sigma\tau}}$$

notwendig und hinreichend

Matrixschreibweise $\quad \boxed{\underline{\underline{\Lambda}}^T \underline{\underline{\eta}}\, \underline{\underline{\Lambda}} = \underline{\underline{\eta}}}$

vgl. Rotationsmatrizen $\quad \underline{\underline{R}}^T \mathbb{1}\, \underline{\underline{R}} = \mathbb{1}$

Lorentzgruppe $\quad LT$ (Liegruppe)

(a) $\quad \Lambda_1, \Lambda_2 \in LT \Rightarrow \Lambda = \Lambda_1 \circ \Lambda_2 \quad , \quad \Lambda^\mu{}_\nu = (\Lambda_1)^\mu{}_\sigma (\Lambda_2)^\sigma{}_\nu$

$\mathbb{1} \in LT \quad$ etc.

Gruppe $\quad$ Untergruppe von reellen 4×4 Matrizen

(b) $\quad \boxed{(\det \Lambda)^2 = 1}$

$\det \Lambda = +1 \quad$ eigentliche Lorentztrafos

(c) $\quad (\sigma = \tau = 0)$

$$\Rightarrow \quad 1 = \eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2$$

$$\boxed{(\Lambda^0{}_0)^2 \geq 1}$$

$\Lambda^0{}_0 \geq 1 \quad$ orthochron

(d) $\quad L_+^\uparrow = \{ \Lambda \in LT : \det \Lambda = 1 , \Lambda^0{}_0 \geq 1 \}$

Untergruppe

(e) $\quad$ Drehungen sind Untergruppe $\quad SO(3) \subset L_+^\uparrow$

$$\Lambda = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array} \right)$$

$$\Lambda^T \eta \Lambda = \eta \quad \Leftrightarrow \quad R^T R = \mathbb{1}$$

(f) Lorentzboost $x'^2 = x^2,\ x'^3 = x^3$

$$(x'^0)^2 - (x'^1)^2 = (x^0)^2 - (x^1)^2 \quad \text{invariant}$$

$$x'^0 = x^0 \cosh\psi - x^1 \sinh\psi$$
$$x'^1 = -x^0 \sinh\psi + x^1 \cosh\psi$$

$$\Lambda(\psi) = \begin{bmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ψ : Rapidität

$$\Lambda(\psi_1)\Lambda(\psi_2) = \Lambda(\psi_1 + \psi_2) \quad \text{1-dim Untergruppe}$$

$$= x'^0 = \cosh\psi\,(x^0 - x^1 \tanh\psi)$$
$$x'^1 = \cosh\psi\,(x^1 - x^0 \tanh\psi)$$

$x'^1 = 0$ falls Beobachter mit konstanter Geschwindigkeit v

$$\Rightarrow \boxed{\tanh\psi = v/c} =: \beta \qquad \tanh^2\psi = \frac{\sinh^2\psi}{\cosh^2\psi} = 1 - \frac{1}{\cosh^2\psi}$$

$$\cosh\psi = \frac{1}{\sqrt{1 - v^2/c^2}} =: \gamma$$

nichtrelativistischer Grenzfall

$$\boxed{\begin{aligned} t' &= t \\ x' &= x - vt \end{aligned}}$$

Galilei-Trafo

$$\boxed{\begin{aligned} t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \\ x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \end{aligned}} \begin{aligned} &= \gamma\,(t - \beta x/c^2) \\ &= \gamma\,(x - \beta c t) \end{aligned}$$

10.2. Tensorbegriff, Vierervektoren

analog zu euklidischen Tensoren $\qquad K'_j = D_{jk} K_k$

<u>Definition</u> 4-er Vektor $V^\mu = (V^0, \vec{V})$ Kontravariant

mit Transformationsgesetz

$$\boxed{V'^\mu = \Lambda^\mu{}_\nu V^\nu} \qquad \text{mit } \Lambda \in LT$$

Tensoren Kontravariant $\boxed{T'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau T^{\sigma\tau}}$ etc.

Folgerung: inneres Produkt invariant

$$V^\mu \eta_{\mu\nu} W^\nu = V^0 W^0 - V^1 W^1 - V^2 W^2 - V^3 W^3$$

$$V'^\mu \eta_{\mu\nu} W'^\nu = \Lambda^\mu{}_\sigma V^\sigma \eta_{\mu\nu} \Lambda^\nu{}_\tau W^\tau = V^\sigma \eta_{\sigma\tau} W^\tau$$

in allen Inertialsystemen

<u>Definition</u> Kovarianter Vektor

$$\boxed{V_\mu \equiv \eta_{\mu\nu} V^\nu}$$

Ziehen von Indices mit Minkowski Tensor

Transformationsgesetz

$$V'_\mu = \eta_{\mu\nu} V'^\nu = \eta_{\mu\nu} \Lambda^\nu{}_\sigma V^\sigma$$

$$V'_\mu \Lambda^\mu{}_\tau = \Lambda^\mu{}_\tau \eta_{\mu\nu} \Lambda^\nu{}_\sigma V^\sigma = \eta_{\tau\sigma} V^\sigma = V_\tau$$

$$\boxed{V'_\mu = (\Lambda^{-1})_\mu{}^\tau V_\tau}$$

usw. für Tensoren

Hinaufziehen von Indizes

$$V_\mu = \eta_{\mu\nu} V^\nu \implies V^\nu = \eta^{\nu\mu} V_\mu$$

$$\eta^{\mu\sigma} \eta_{\sigma\nu} = \delta^\mu{}_\nu$$

Kronecker

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}, \quad \boxed{V_\mu = (V^0, -\vec{V})}$$

<u>Tensorfelder</u>

$$S'(x') = S(x) \qquad \text{Skalar}$$

$$V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x) \qquad \text{Vektor, etc}$$

$$\partial_\mu S(x) \equiv \frac{\partial S}{\partial x^\mu}(x) \quad \text{transformiert sich kovariant!}$$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu{}_\mu \partial_\nu$$

$$\partial_\mu = (\partial_0, \vec{\nabla})$$

Wellenoperator, d'Alembert

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$$= \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial^\nu \partial_\nu$$

$$= \partial'^\nu \partial'_\nu \qquad \text{invariant unter Lorentz-Trafo}$$

Wellengleichung

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\psi = 0$$ invariant

Kontinuitätsgleichung

$$\partial_t \rho + \vec{\nabla}\cdot\vec{J} = 0$$

Vierervektor $J^\mu = (c\rho, \vec{J})$

$$\boxed{\partial_\mu J^\mu = 0}$$ invariant

$$\boxed{\begin{aligned} &\vec{A} \to \vec{A} + \vec{\nabla}\chi \\ &\varphi \to \varphi - \tfrac{1}{c}\partial_t \chi \\ &\text{Umeichen} \\ &A^\mu \longrightarrow A^\mu + \partial^\mu \chi \end{aligned}}$$

Lorentzeichung

$$\frac{1}{c}\partial_t \varphi + \vec{\nabla}\cdot\vec{A} = 0$$

Vierervektor $A^\mu = (\varphi, \vec{A})$ $\boxed{\partial_\mu A^\mu = 0}$

Wellengleichungen

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{A} = \frac{4\pi}{c}\vec{J} - \vec{\nabla}\left(\frac{1}{c}\partial_t\varphi + \vec{\nabla}\cdot\vec{A}\right)$$

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\varphi = 4\pi\rho + \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\varphi + \nabla\cdot\vec{A}\right)$$

$$\boxed{\Box A^\mu = \frac{4\pi}{c}J^\mu + \partial^\mu(\partial_\nu A^\nu)}$$

Ladungserhaltung $\Longrightarrow$ $\boxed{\partial_\mu J^\mu = 0}$

Lorentz-Eichung $\boxed{\Box A^\mu = \frac{4\pi}{c}J^\mu}$

elektrisches Feld

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\varphi$$

magnetisches Feld

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

elektromagnetischer Feldstärke - Tensor, Faraday - Tensor

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$ antisymmetrisch $F_{\mu\nu} = -F_{\nu\mu}$

eichinvariant $\boxed{A_\mu \to A_\mu + \partial_\mu \chi}$

$$F_{10} = \partial_1 A_0 - \partial_0 A_1 = \partial_x \varphi + \frac{1}{c}\partial_t A_x = -E_x$$

$$F_{20} = -E_y \ , \quad F_{30} = -E_z$$

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = -\partial_x A_y + \partial_y A_x = -B_z$$

$$F_{23} = -B_x \ , \quad F_{31} = -B_y$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\tau}F_{\sigma\tau} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & +B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{4\pi}{c} J^\nu$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$$

inhomogene Maxwellgl.
Kovariant

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}\vec{J}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Potentialbedingungen

$$\boxed{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0}$$

Lorentz - Trafo

$$F'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau F^{\sigma\tau},$$

Lorentz - Boost

$$\Rightarrow \boxed{\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \beta B_z) & B_y' &= \gamma(B_y + \beta E_z) \\ E_z' &= \gamma(E_z + \beta B_y) & B_z' &= \gamma(B_z - \beta E_y) \end{aligned}}$$

Levi - Civita - Symbol

$\epsilon^{\mu\nu\sigma\tau}$ total antisymmetrisch

$$\epsilon^{\mu\nu\sigma\tau} = -\epsilon^{\nu\mu\sigma\tau} = -\epsilon^{\sigma\nu\mu\tau} = -\epsilon^{\tau\nu\sigma\mu}$$

$\epsilon^{0123} := 1$ nur ein Wert

Pseudo - Tensor invariant

$$\epsilon'^{\alpha\beta\gamma\delta} = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\Lambda^{\gamma}{}_{\sigma}\Lambda^{\delta}{}_{\tau}\,\epsilon^{\mu\nu\sigma\tau}$$

total antisymmetrisch

$$\epsilon'^{0123} = \Lambda^{0}{}_{\mu}\Lambda^{1}{}_{\nu}\Lambda^{2}{}_{\sigma}\Lambda^{3}{}_{\tau}\,\epsilon^{\mu\nu\sigma\tau}$$

$$= (\det \Lambda) = \pm 1$$

Potentialbedingungen

$$\boxed{\epsilon^{\mu\nu\sigma\tau}\partial_{\sigma}F_{\mu\nu} = 0}$$

Invariante Skalar

$$\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \ldots = \vec{B}^2 - \vec{E}^2$$

unabhängig vom Bezugssystem

$$\frac{1}{2}\epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau} = \vec{B}\cdot\vec{E}$$ invariant