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T II: Elektrodynamik

(Prof. E. Frey)

Problem set 4

Tutorial 4.1 *Superconductor*

The constitutive equation of a type-I superconductor relates the the supercurrent density $\vec{J}_s$ directly to the vector potential $\vec{A}$ via the second London equation,

$$\vec{J}_s(\vec{x}) = -\frac{n_s e^2}{mc}\vec{A}(\vec{x})\,.$$

Here m and $-e$ denote the mass and charge of the supercurrent carrier, and n_s abbreviates their number density.

a) Use Maxwell's equations to show that in the static case the magnetic field fulfills the field equation

$$\nabla^2\vec{B}(\vec{x}) - \frac{1}{\lambda_L^2}\vec{B}(\vec{x}) = 0\,,$$

and determine the *London penetration depth* λ_L. Conclude that no homogeneous magnetic field can exist in the bulk of a superconductor.

b) Consider the boundary $z = 0$ between a superconductor $(z > 0)$ and vacuum $(z < 0)$. A magnetic field $\vec{B}$ is applied parallel to the boundary $(z < 0)$. Solve for the magnetic field inside of the superconductor.

c) Show that the field equation can be obtained by minimizing the total energy $U = U_{\text{matter}} + U_{\text{field}}$ by varying with respect to the vector potential, $\vec{A}(\vec{x}) \to \vec{A}(\vec{x}) + \delta\vec{A}(\vec{x})$. Here the variation of the matter and field energy follows from

$$\delta U_{\text{matter}} = -\frac{1}{c}\int \mathrm{d}^3\vec{x}\,\vec{J}_s(\vec{x})\cdot\delta\vec{A}(\vec{x}) \qquad \text{and} \qquad \delta U_{\text{field}} = \frac{1}{4\pi}\int \mathrm{d}^3\vec{x}\,\vec{B}(\vec{x})\cdot\delta\vec{B}(\vec{x})\,.$$

The supercurrent $J_s(\vec{x})$ and the magnetic field $\vec{B}(\vec{x})$ have to be eliminated in favor of $\vec{A}(\vec{x})$ to perform the variation.

Tutorial 4.2 *Polarizable Medium*

Consider the following constitutive equation for a polarizable medium

$$\partial_t^2 \vec{P}(\vec{x},t) + \frac{1}{\tau}\partial_t \vec{P}(\vec{x},t) + \omega_0^2 \vec{P}(\vec{x},t) = \frac{\omega_p^2}{4\pi}\vec{E}(\vec{x},t)\,;$$

here $\omega_p^2 = 4\pi n e^2/m$ denotes the plasma frequency, ω_0^2 the resonance frequency, and $\tau > 0$ is a characteristic damping time. The change of polarization is considered slow and possibly induced magnetic fields shall be neglected.

a) Employing the constitutive equation, evaluate the time derivative $\dot{u}_M(\vec{x},t)$ of the energy density of matter

$$u_M(\vec{x},t) = \frac{2\pi}{\omega_p^2}\left(\vec{j}^{(\mathrm{ind})}(\vec{x},t)^2 + \omega_0^2 \vec{P}(\vec{x},t)^2\right),$$

and interpret the terms contributing to $\dot{u}_M(\vec{x},t)$.

b) Consider the total energy density $u = u_M + u_F$, with the usual field energy density $u_F = (1/8\pi)[\vec{E}^2 + \vec{B}^2]$, derive a local balance equation

$$\partial_t u(\vec{x},t) + \operatorname{div} \vec{S}(\vec{x},t) = q(\vec{x},t) \quad \text{where} \quad \vec{S}(\vec{x},t) = \frac{c}{4\pi}\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)\,,$$

and determine the source term $q(\vec{x},t)$.

Problem 4.3 *Debye-Hückel theory*

In an electrolyte solution ions of opposite charges can freely float agitated by thermal fluctuations. To simplify consider only one species of cations of charge $q_+ > 0$ and anions of charge $q_- < 0$ of respective number density n_+ and n_-. Charge neutrality requires $q_+ n_+ = q_- n_-$. The constitutive equation of the Debye-Hückel electrolyte relates the charge densities of the cations $\rho_+(\vec{x})$ and anions $\rho_-(\vec{x})$ to the electrostatic potential $\varphi(\vec{x})$ via

$$\rho_+(\vec{x}) = q_+ n_+ \exp\left(-\frac{q_+\varphi(\vec{x})}{k_B T}\right), \qquad \rho_-(\vec{x}) = q_- n_- \exp\left(-\frac{q_-\varphi(\vec{x})}{k_B T}\right).$$

Here T denotes the temperature and k_B is Boltzmann's constant.

a) Considering external charges $\rho^{\mathrm{ext}}(\vec{x})$ in addition to the induced ones, $\rho^{\mathrm{ind}}(\vec{x}) = \rho_+(\vec{x}) + \rho_-(\vec{x})$, formulate the Poisson equation. Linearize the exponentials in $\varphi(\vec{x})$ and show that

$$\nabla^2\varphi(\vec{x}) - \frac{1}{\lambda^2}\varphi(\vec{x}) = -4\pi\rho^{\mathrm{ext}}(\vec{x})\,,$$

holds. Relate the Debye-Hückel *screening length* λ to the number densities $n_\pm$.

b) Determine the electrostatic potential within the linearized theory for a point-like test charge $\rho^{\mathrm{ext}}(\vec{x}) = Q\delta(\vec{x})$. Use the spherical symmetry of the problem and determine the solution of the differential equation for $r = |\vec{x}| > 0$ that vanishes for $r \to \infty$. Discuss the physical consequences of the result.

Hint: The substitution $\varphi(r) = u(r)/r$ simplifies the homogeneous differential equation. The constant of integration may be determined by matching to the Coulomb solution in vacuum close to the test charge.

c) Equivalently you may evaluate the Green function $G(\vec{x},\vec{y})$ defined via

$$\left(\frac{1}{\lambda^2} - \nabla^2\right) G(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y}).$$

Show that

$$G(\vec{k}) = \int d^3\vec{x}\ e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} G(\vec{x},\vec{y})\,.$$

satisfies an algebraic equation and perform the inverse Fourier transform; apply the residue theorem to perform the integration.

d*) Consider a small spherical colloid of radius R suspended in the electrolyte. The colloid carries a charge Ze homogeneously distributed along the surface. There are no further charges inside of the colloid. Since the electrolyte cannot penetrate the colloidal particles, the usual Poisson equation holds in the inner region. Determine the electrostratic potential and compare your result to the point-like test charge.

Problem 4.4 *Rotating sphere*

A sphere S of constant surface charge density σ rotates at constant angular frequency $\vec{\omega}$, i.e., there is a surface current density $\vec{K}(\vec{r}) = \sigma\vec{v}(\vec{r})$ for $\vec{r} \in \partial S$, where $\vec{v}(\vec{r}) = \vec{\omega} \times \vec{r}$ abbreviates the velocity of the point $\vec{r}$.

a) Calculate the corresponding magnetic field $\vec{B}(r)$ inside and outside of the sphere. It is favorable to determine first a suitable vector potential by evaluating the analogue of Coulomb's solution,

$$\vec{A}(\vec{r}) = \frac{1}{c}\int_{\partial S} \frac{\vec{K}(\vec{R})}{|\vec{r}-\vec{R}|}\, dS(\vec{R})\,.$$

The integral over the surface of the sphere ∂S is elementary in polar coordinates aligned with $\vec{r}$ once the polar angle ϑ is eliminated in favor of the distance $s = |\vec{r}-\vec{R}| = \sqrt{r^2 + R^2 - 2rR\cos\vartheta}$.

b) In a minimal model, the origin of the magnetic field of the earth is attributed to the rotating molten and ionized core of the earth. (Dynamo theory explains why it rotates.) Calculate the magnetic dipole field outside a homogeneous, rotating charge distribution, $\rho(\vec{r}) = \rho_0$ for $r < R_S$.

Note: The surface charge density of a shell of width dR is related to the charge density ρ_0 by $\sigma = \rho_0 dR$.

Problem 4.5 *Spherical capacitor*

A spherical capacitor is composed of two concentric, metallic spheres of radii $R_1 < R_2$, the region between the spheres being filled with an insulator of dielectric constant ε. While the outer sphere is kept uncharged, the inner sphere is charged such that the potential difference between inner and outer sphere adjusts to U.

a) Determine the electrostatic potential $\varphi(\vec{r})$ of the problem that vanishes at infinity.

b) Calculate the charge Q on the inner sphere; substitute U in favor of Q in your result for $\varphi(\vec{r})$. What is the capacity of the assembly?

c) Derive the electric field $\vec{E}(\vec{r})$ by means of heuristic arguments. Check that your result agrees with the one obtained from $\varphi(\vec{r})$.

Due date: 5/22/07