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T II: Elektrodynamik

(Prof. E. Frey)

Problem set 0

Tutorial 0.1 *Charged particle with $\vec{E} \times \vec{B}$ drift*

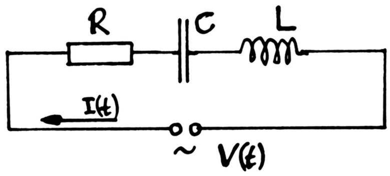
A point particle with charge e is moving in a static electromagnetic field, i. e. the equation of motion reads,

$$m\dot{\vec{v}} = e\left(\vec{E} + \frac{1}{c}\,\vec{v} \times \vec{B}\right).$$

Calculate the velocity $\vec{v}(t)$ and the trajectory $\vec{r}(t)$ of the particle. Show that the fields impose a constant drift term on the velocity being proportional to $\vec{E} \times \vec{B}$. For the case $E_z = 0$, give a sketch of the trajectory of the particle and discuss the different cases of this cycloid motion. Finally, show explicitly that the energy gain of the particle vanishes on average.

Choose the coordinate frame such that the magnetic field defines the z-axis, $\vec{B} = B\hat{e}_z$; then, the motion along the z-axis separates. The remaining two coupled differential equations may be solved by introducing a complex velocity, $\zeta := v_x + \mathrm{i}v_y$, and a complex field, $\mathcal{E} := E_x + \mathrm{i}E_y$.

Tutorial 0.2 *LRC circuit I*



An electric circuit consisting of a coil with self-inductance L, a resistor with resistance R, and a capacitor with capacitance C is driven by the external voltage $V(t)$.

a) Relying on Kirchhoff's laws, argue that the charge on the capacitor $Q(t)$ fulfills the second order differential equation

$$L\ddot{Q}(t) + R\dot{Q}(t) + \frac{1}{C}Q(t) = V(t).$$

b) First consider $V(t) \equiv 0$. Determine the charge $Q(t)$ for the initial conditions $Q(t=0) = Q_0$, $\dot{Q}(t=0) = I_0$. Discriminate the cases of small and large damping.

Problem 0.3 *LRC circuit II*

This problem continues Tutorial 0.2.

c) Determine the response $Q(t)$ due to a short voltage pulse, $V(t) = \Phi\delta(t)$, assuming $Q(t<0) \equiv 0$. To obtain the solution, rewrite the second order differential equation into a set of two coupled first order equations

$$\dot{Q}(t) - I(t) = 0\,, \qquad L\dot{I}(t) + RI(t) + \frac{1}{C}Q(t) = \Phi\delta(t).$$

For times $t > 0$, one recovers the homogeneous equations with corresponding solutions derived above. By integrating over a small time interval, $-\varepsilon \le t \le \varepsilon$ ($\varepsilon \downarrow 0$) show that the pulse induces a current $I(t = 0^+) = \Phi/L$, but no charge, $Q(t = 0^+) = 0$, which serve as initial conditions.

d) A general time-dependent applied voltage may be represented as a linear superposition of pulses, $V(t) = \int d\bar{t}\, V(\bar{t})\delta(t-\bar{t})$, where $\delta(t)$ denotes Dirac's delta function. Show that the general solution for the charge in the limit of weak damping is given by

$$Q(t) = \int_{-\infty}^{\infty} d\bar{t}\, \chi(t-\bar{t})\, V(\bar{t}) \qquad \text{where} \qquad \chi(t) = \Theta(t) e^{-\gamma t/2} \frac{\sin(\omega_r t)}{\omega_r L},$$

with suitably chosen γ and ω_r, and $\Theta(t)$ denotes the Heaviside step function,

$$\Theta(t) = \begin{cases} 1 & \text{for } t \ge 0, \\ 0 & \text{else.} \end{cases}$$

You may either rely on the superposition principle and use the previous considerations or prove the relation by direct substitution.

e) Perform a Fourier transform of $\chi(t)$, convention $\chi_\omega = \int_{-\infty}^{\infty} \chi(t)\, e^{i\omega t}\, dt$, and determine the frequency-dependent complex susceptibility χ_ω. By the convolution theorem, the charge response in the frequency domain is related to the external driving by $Q_\omega = \chi_\omega V_\omega$. Show that χ_ω may be obtained much easier by a Fourier transform of the differential equation. Determine the real part χ'_ω and imaginary part χ''_ω of $\chi_\omega = \chi'_\omega + i\chi''_\omega$ and show that

$$\chi'_\omega = \int_{-\infty}^{\infty} \chi(t) \cos(\omega t) dt\,, \qquad \chi''_\omega = \int_{-\infty}^{\infty} \chi(t) \sin(\omega t) dt\,.$$

Sketch χ'_ω and χ''_ω as a function of frequency ω.

f) Determine the current $I(t)$ for harmonic driving, $V(t) = \mathrm{Re}(V_\omega e^{-i\omega t})$. Calculate the dissipated power of the circuit and state its connection to χ''_ω.

Problem 0.4 *Charging energy*

The electrostatic interaction energy of a continuous charge distribution may be calculated as a double volume integral,

$$U = \frac{1}{2} \int d^3\vec{x}\, d^3\vec{y}\, \frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x} - \vec{y}|}\,.$$

a) Argue that this expression is the proper generalization of the well-known interaction energy $Q_1 Q_2 / r$ of two point charges Q_1, Q_2 separated by a distance r.

b) Evaluate the integral expression for the charging energy U for the case of a homogeneously charged sphere of radius R and total charge Q. Perform first the $\vec{y}$-integral for fixed $\vec{x}$ introducing spherical coordinates where the direction of the north pole is aligned with the $\vec{x}$-axis. The integration over the polar angle $\vartheta = \angle(\vec{x}, \vec{y})$ is elementary, if ϑ is eliminated in favor of the distance $s = |\vec{x} - \vec{y}| = \sqrt{x^2 + y^2 - 2xy\cos\vartheta}$.

Due date: Tuesday, 4/24/07, at 9 a.m.