Lecture 5: ppt.

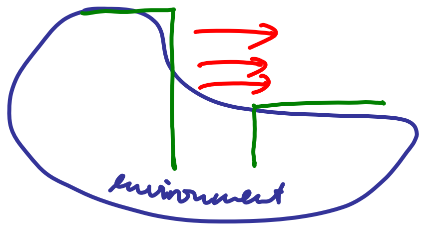
Lecture 6: Scattering Bethe Ansatz (SBA)

If leads are good thermal bath ($L \xrightarrow[\delta \to 0]{} \infty$), sufficient dissipation at $\pm\infty$

then: system out of equilibrium can be described by a single eigenstate $|\psi\rangle_S$ of $H = H_0 + H_1$

$|\psi_S\rangle$ - determined by BC set by leads.

- describes all local steady-state expectation values
- describes entropy production (of open system as subpart of the "world") and dissipation



Why? Eigenstates of open system,

information disappears at $\pm\infty$!!

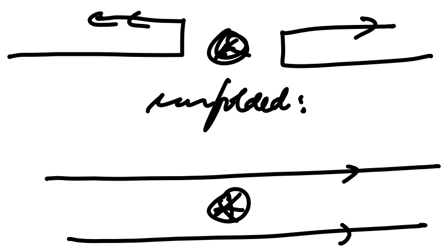
$|\psi_S\rangle$ satisfies Lippmann-Schwinger:

$$|\psi\rangle_S = |\varphi\rangle = \frac{1}{E - H_0 + i\eta} H_1 |\psi\rangle$$

BC: $\psi(x_1, \ldots x_N) \xrightarrow[x_i \to -\infty]{} \varphi_{bath}(x_1, \ldots, x_N) = \prod_j^{\mu_1} e^{i x_j k_j^{(1)}} \prod_j^{\mu_2} e^{i x_\ell K_\ell^{(2)}}$

Construct scattering eigenstates (SES) via scattering Bethe Ansatz (SBA)

Interacting resonance level model (IRL)



$$H_{IRL} = \underbrace{-i\int dx \left[\psi_1^\dagger \partial_x \psi_1 + \psi_2^\dagger \partial_x \psi_2 + \varepsilon_d d^\dagger d\right] + t(\psi_1^\dagger + \psi_2^\dagger)d + h.c.}_{RL}$$
$$+ U d^\dagger d (\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2)$$

BAnsatz: $H = -i\int dx\, \psi^\dagger \partial_x \psi + \varepsilon_d\, d^\dagger d + t\left(\psi^\dagger(0)\, d + d^\dagger \psi(0)\right)$

① ② ③ ③†

$$N = \int dx\, \psi^\dagger\psi + d^\dagger d$$

N_1 : $|1\rangle = \Big[\int dx\, \underset{\alpha}{F(x)\,\psi^\dagger(x)} + \underset{\beta}{g\, d^\dagger}\Big]|0\rangle$ most general $N=1$ state.

Demand: $H|1\rangle = E|1\rangle \Rightarrow$ fixes $F(x)$, g

①α $\quad -i\int\left(\partial_x F(x)\right)\psi^\dagger(x) + \varepsilon_d\, g\, d^\dagger + t\, F(0)\, d^\dagger + t\, g\, \psi^\dagger(0)$

$$= E\Big[\int dx\, F(x)\,\psi^\dagger(x) + g\, d^\dagger\Big]|0\rangle$$

Collect $\psi^\dagger$ terms:
$$\boxed{\begin{aligned} -i\partial_x F + t\,g\,\delta(x) &= E\,F(x) \\ \varepsilon_d\, g + t\, F(0) &= E\, g \end{aligned}}$$
Schrödinger Eq.

WF gives divergencies

in field theory, this means that $(\psi^\dagger\psi)^2$ have to be regularized.

∂_x is many-body theory in disguise:

it comes from linearizing spectrum around Fermi surface of filled Fermi sea!! If you fill it, you get many body field theory with smooth properties.

$$\delta(x)\,\theta(x) = \tfrac{1}{2}\delta(x)$$

Solve: $g = \dfrac{t}{E - \varepsilon_d} F(0)$

$$-i\partial F + \frac{t^2}{E-\varepsilon_d} F(0)\,\delta(x) = E\,F(x)$$

phase shift depends on g; i.e. on k

$F = e^{ikx}\left[A\,\theta(-x) + B\,\theta(x)\right]$, $\quad E = k$

$S = B/A$ = phase

$-iF = kF + \delta(x)(-i)\dfrac{A-B}{2} + \overbrace{\dfrac{t^2}{k-\varepsilon_d}}^{g}\left(\dfrac{A+B}{2}\right)$ $\quad \boxed{\varphi = 2\tan^{-1} g}$

$(-i+g)A = (-i-g)B$, $\quad S = \dfrac{B}{A} = \dfrac{i - \frac{t^2}{k-\varepsilon_d}}{1 + {t^2}/{k-\varepsilon_d}} = e^{i\varphi(k)}$

Regularization: $:(\psi^\dagger\psi)^2:$ regularize by
- point-splitting
- Pauli-Villars
- dimensionless regularization
- finite length
- finite bandwidth

Solve in different way:

$$-i\partial F + g\,\delta(x)\,F = E\,F$$

$$\partial_x(\log F) = \frac{F'}{F} = i(E - g\,\delta(x))$$

$$\ln F = i\int^x dx'\,(E - g(x)) = iEx - \frac{ig}{2}\,\underbrace{\varepsilon(x)}_{=\mathrm{sign}(x)}$$

$$F = e^{iEx}\,e^{-\frac{ig}{2}\varepsilon(x)}$$

$$= e^{iEx}\left[\theta(-x)e^{ig/2} + e^{-ig/2}\,\theta(x)\right]$$

$$\Rightarrow \quad B/A = e^{-ig} \Rightarrow \varphi = g$$

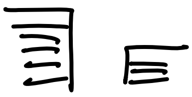
$$F = e^{ikx}\left[A\,\theta(-x) + e^{i\varphi_k}\,\theta(x)\right] \qquad \varphi_k = \frac{t^2}{k-\varepsilon_d}$$

$$G = \frac{t\,F(0)}{k-\varepsilon_d} = \frac{t}{k-\varepsilon_d}\,A\frac{(1+e^{i\varphi})}{2}$$

$$|F\rangle = A\int dx\left\{e^{ikx}\left[\theta(-x) + e^{i\varphi_k}\,\theta(x)\right] + G\,d^\dagger\right\}|0\rangle \equiv \int \alpha_k^\dagger(x)\,|0\rangle$$

since H is quadratic, we can write

$$|F\rangle_N = \int dx_1 \dots \int dx_N\ \alpha^\dagger_{k_1}(x) \dots \alpha^\dagger_{k_N}(x)\ |0\rangle$$

<u>two leads:</u>  RLM.

$$H = -i\int dx\,(\psi_1^\dagger\partial_x\psi_1 + \psi_2^\dagger\partial_x\psi_2)\,dx + \varepsilon_d\,d^\dagger d + t\,d^\dagger(\psi_1+\psi_2) + h.c.$$

$$\psi_e = (\psi_1+\psi_2)/\sqrt{2}$$
$$\psi_o = (\psi_1-\psi_2)/\sqrt{2}$$

$$= \left.\begin{array}{l} -i\int dx\,\psi_e^\dagger\partial\psi_e + \varepsilon_d\,d^\dagger d + t\,d^\dagger\psi_e + h.c. \\ -i\int dx\ \psi_o^\dagger\partial_x\psi_o \end{array}\right\}\ \text{was solved above}$$

Most general solution for $N=1$:

$$|1\rangle = \left[\int F\,\psi_e^\dagger\,dx + G d^\dagger + \int H(x)\,\psi_0^\dagger(x)\right]|0\rangle$$

$$\hat{H}|1\rangle = k|1\rangle \qquad\qquad \downarrow \; -i\partial H = kH\,, \quad H = Be^{ikx}$$

k for H can be chosen same as for G, because in both cases k describes the incoming momenta [with periodic b.c. this would not be the case; here we can do it, because of $L\to\infty$]

$$|1\rangle = e^{ikx}\left\{A\left[\theta(-x)+e^{i\varphi}\theta(x)\right]\psi_e^\dagger(x) + B\,\psi_0^\dagger(x) + Gd^\dagger\right\}|0\rangle$$

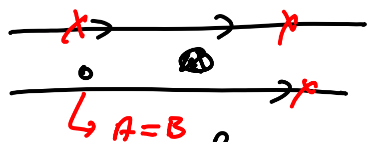
to impose BC, go back to 1,2 basis:

$$= e^{ikx}\left\{A\left[\theta(-x)+e^{i\varphi}\theta(x)\right](\psi_1^\dagger+\psi_2^\dagger)/\sqrt{2} + \frac{B}{\sqrt{2}}\left(\theta(x)+\theta(-x)\right)(\psi_1^\dagger-\psi_2^\dagger) + Gd^\dagger\right\}|0\rangle$$

$$= \frac{1}{\sqrt{2}}e^{ikx}\Big\{\left[(A+B)\theta(-x) + (Ae^{i\varphi}+B)\theta(x)\right]\psi_1^\dagger + \left[(A-B)\theta(-x) + (Ae^{i\varphi}-B)\theta(x)\right]\psi_2^\dagger + Gd^\dagger\Big\}|0\rangle$$

Now impose boundary conditions:

$|1k\rangle$

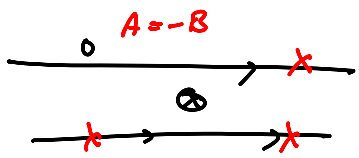


$$|1k\rangle = A\sqrt{2}\int dx\, e^{ikx}\left[\theta(-x) + \frac{1+e^{i\varphi}}{2}\theta(x)\right]\psi_1^\dagger(x) + \overbrace{\frac{e^{i\varphi}+1}{2}}^{R}\theta(x)\,\psi_1^\dagger(x)$$
$$+ \overbrace{\frac{e^{i\varphi}-1}{2}}^{T}\theta(x)\,\psi_2^\dagger(x) + Gd^\dagger|0\rangle$$

$$= \int dx\; \alpha_{1k}^\dagger(x)\,|0\rangle$$

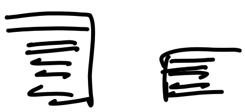
Similarly, for lead 2:

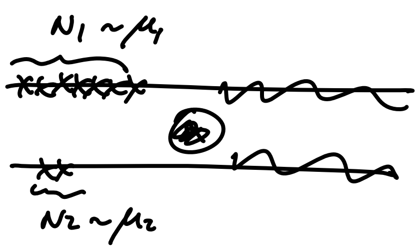
$|2k\rangle$



$$|2k\rangle = \int dx\; \alpha_{2k}^\dagger(x)\,|0\rangle$$

Multiparticle:





$$|N_1,N_2\rangle = \int \prod dx_j \prod dy_\ell \;\prod \alpha_{1k_j}^\dagger(x_j)\;\prod \alpha_{2p_\ell}^\dagger(y_\ell)\;|0\rangle$$

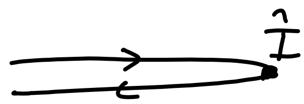
Expectation values: calculate in $L \to \infty$, then single-particle state become orthonormal.

$$\frac{\langle \mu_1, \mu_2 | I | \mu_1, \mu_2 \rangle}{\langle \mu_1, \mu_2 | \mu_1, \mu_2 \rangle} = \sum \frac{t^2}{(k_1 - \varepsilon_d)^2 + t^2} - \sum \frac{t^2}{(k_2 - \varepsilon_d)^2 + t^2}$$

$$= \int dk \left[\frac{t^2}{(k_1 + \varepsilon_d)^2 + t^2} f_1(k_1) - \frac{t^2 f_2(k_2)}{(k_2 - \varepsilon_d)^2 + t^2} \right]$$

= Keldysh result!

$$\langle \varphi | U^\dagger(0, -\infty) \, I \, \underbrace{U(0, \infty)}_{T \, e^{-\int i t H'(t')}} | \varphi \rangle$$



$$\frac{\langle \mu_1, \mu_2 | n | \mu_1, \mu_2 \rangle}{\langle \mu_1 \mu_2 | \mu_1 \mu_2 \rangle} =$$

If $L \to \infty$:
$$\langle 1p | 1p' \rangle = \delta_{pp'}$$
$$\langle 1p | 2p \rangle = 0$$

Interacting Resonant Level Model (Thesis of P. Metha)

$$H = H_{quad} + 2 U (\psi_1^\dagger \psi_1) d^\dagger d \quad , \quad \psi_{e/o} = \frac{1}{\sqrt{2}} (\psi_1 \pm \psi_2)$$
$$= H_e + H_o$$

$$H_e = -i \int dx \, \psi_e^\dagger \partial_x \psi_e + t \, \delta(x) \left[\psi_e^\dagger(0) d + h.c. \right] + \varepsilon_d d^\dagger d + U \psi_e^\dagger \psi_e \underline{\underline{d^\dagger d}}$$

$$H_o = -i \int dx \, \psi_o^\dagger \partial_x \psi_o + U \psi_o^\dagger \psi_o \underline{\underline{d^\dagger d}}$$

interactions make it complicated!

$\simeq$ fluctuating potential

Construct single-particle eigenfunctions of H:

$$\Big[A \Big(\int dx \, \overset{F}{\overset{\downarrow}{g_p(x)}} \, \psi_e^\dagger(x) + e_p d^\dagger \Big) + B \int dx \, h_p(x) \, \psi_o^\dagger(x) \Big] |0\rangle$$

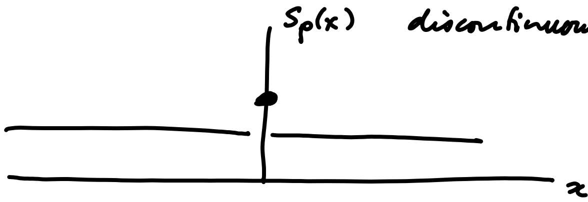
$$g_p(x) = \frac{e^{ipx}}{1 + e^{i\delta_p}} \left[\theta(-x) + e^{i\delta_p} \theta(x) \right] \quad , \quad g_p(0) = 1$$

$$\delta_p = 2 \tan^{-1} \frac{t^2}{2(p - \varepsilon_d)}$$

In presence of interactions, $h^{\pm}$ should be discontinuous.

$$h_p^{\pm}(x) = \begin{cases} \dfrac{2}{1+e^{i\delta_p}} \, e^{ipx} & x \neq 0 \\ \underbrace{\pm 1}_{g_p(x)} \, e^{ipx} & x = 0 \end{cases}$$

$$= e^{ipx} \, s_p(x)$$



$$\frac{\partial s}{\partial x} = -\delta(-a) + \delta(a) \xrightarrow[a \to 0]{} 0$$

$$|1p\rangle = \int dx \, e^{ipx} \, \alpha^{+}_{1p}(x) \, |0\rangle$$

$$\hookrightarrow = g_p(x) \, \psi_e^{+}(x) + h_p^{+}(x) \, \psi_o^{+}(x) + e_p \, d^{+} \delta(x)$$

convenient choice

$$|2p\rangle = \int dx \, e^{ipx} \, \alpha^{+}_{2p}(x) \, |0\rangle$$

$$\hookrightarrow = g_p(x) \, \psi_e^{+}(x) - h_p^{-}(x) \, \psi_o^{+} + e_d \, d^{+} \delta(x)$$

Most general 2-particle state:

$$|2\rangle \int dx_1 \, dx_2 \Big[A \, g(x_1, x_2) \, \psi_e^{+}(x_1) \psi_e^{+}(x_2) + c \, h(x_1, x_2) \, \psi_o^{+}(x_1) \, \psi_o^{+}(x_2)$$

$$+ B \, j(x_1, x_2) \, \psi_e^{+}(x_1) \psi_o^{+}(x_2) + \int dx \Big[A \, e(x) \, \psi_e^{+}(x) d^{+} + B \, f(x) \, \psi_o^{+}(x) d^{+} \Big] |0\rangle$$

$H|2\rangle = E|2\rangle \Rightarrow$ five equations:

$$(-i\partial_1 - i\partial_2 - E) \, g(x_1, x_2) - \frac{t}{2} \left[\delta(x_1) \, e(x_2) - \delta(x_2) \, e(x_1) \right] = 0$$

$$(-i\partial_1 - i\partial_2 - E) \, h(x_1, x_2) = 0$$

$$(-i\partial_1 - i\partial_2 - E) \, j(x_1, x_2) - t \, \delta(x_1) \, f(x_2) = 0$$

$$(-i\partial_x - E + \varepsilon_d) \, f(x) - t \, j(0, x) + U \, \delta(x) \, f(x) = 0$$

$$(-i\partial_x - E + \varepsilon_d) \, e(x) - 2t \, g(0, x) + U \, \delta(x) \, e(x) = 0$$

We'll see that solution can be constructed from single-particle solutions.

Integrability: protected by conservation laws!

no precession

Hydrogen atom: conserved quantities: $-\partial^2 + \frac{e^2}{r}$, L_i, $\vec{L}\times\vec{p}$

dynamic conservation laws

Integrable field theories have infinitely many dynamic conservation laws.

- Kondo, Anderson model must have such laws, which nobody found yet!
- Sine-Gordon,
- Hubbard, Heisenberg, Haldane-Shastry ...
- Non-linear sigma model ...

If the solution is not obvious, it could be

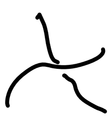
- that you were too stupid to see solution
- a related model exists, with extra irrelevant term, which however makes the model integrable

$\Rightarrow$ Finding integrable models is a 'form of art'

Construct BA for 2-particle wf. in terms of 1-particle wf.

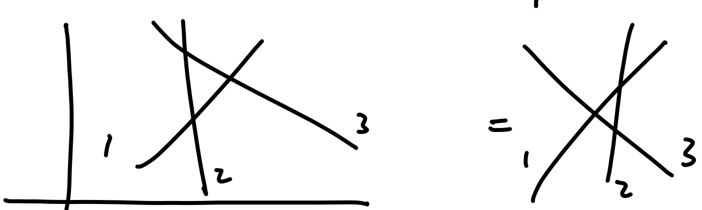
(This does not always work: 1981, Haldane claimed to have solved the Bose-Hubbard
Choy SU(N) " " "

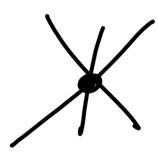
with $S^{ij} = \dfrac{\sin k_i - \sin k_j + ic\,P^{ij}}{\sin k_i - \sin k_j + ic}$ $\quad c\;\varphi^\dagger_{ja},\ a = 1,\dots,N$

Choy: $(P^{ij})^{b_i b_j}_{a_i a_j}$ 

A few months later, Haldane & Choy wrote interesting erratum:

3-particle WF does not satisfy Schr. Eq., when 3 or more particles are on the same site (not possible for SU(2)) $\Rightarrow$ YB is necessary but not sufficient condition!



overlooks possibility that 3 particles can interact. 

On lattice, this matters. In continuum, it does not!!

Construct Ansatz: not seen by $\partial_1 + \partial_2$

$$g(x_1, x_2) = \frac{1}{2}\, g_p(x_1)\, g_k(x_2)\, Z(x_1 - x_2) - g_p(x_2)\, g_k(x_1)\, Z(x_2 - x_1)$$

$\Downarrow$ 5 pages of algebra:

Solution: $\displaystyle\int dx_1 \ldots dx_N \; e^{i\sum p_j x_j} \; \underbrace{F(x_1 \ldots x_N)}_{\text{phase shifts, scattering} \ldots} \; \prod \alpha^{\dagger}_{1 p_i} \; \prod \alpha^{\dagger}_{2 p_j} \; |0\rangle.$

$$F = e^{\,i \sum_{ij} \Phi(p_i, p_j)\, \mathrm{sign}(x_i - x_j)}$$

this works only due to the discontinuities in $h^{\pm}(x)$ at 0.

$$\boxed{S_2^n = \frac{1 - i\varepsilon p}{1 - i\varepsilon} = \Phi_2(p_1, p_2) = U \tan^{-1} \frac{U(p_1 - p_2)}{2(p_1 + p_2 - 2\varepsilon d)}}$$

the same function works independent of whether p_1, p_2 are particle 11, 12, 22.

(complete in limit $N \to \infty$)

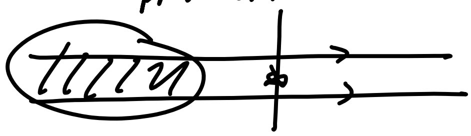
$\uparrow$ the 2-particle S-matrix depends on U!!

You have to prepare the basis properly!!

So, now we have a complete set of N-particle scattering eigenstates of $\hat{H}$

Now consider $\psi(p \ldots k) = \langle x_1 \ldots x_N | p \ldots k \rangle$

prescribed!



$$\alpha_2(x) = \left[\theta(-x) + e^{i\delta}\, \theta(x)\right] \varphi_2 + \theta(x)\, \varphi_1 + \delta(x)\, d$$

want: product of plane waves.

For $x_j < 0$:

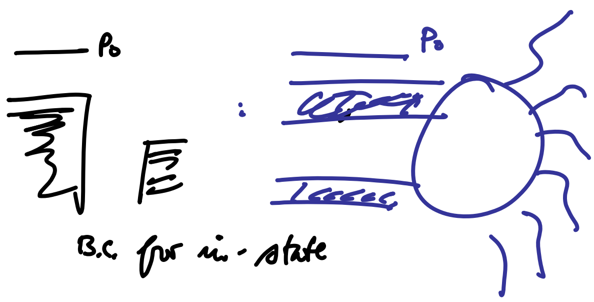
$$|\{p\}\rangle_0 = e^{i\sum x_j p_j}\; e^{\,i \sum_{ij} \Phi(p_i, p_j)\, \mathrm{sign}(x_i - x_j)} \prod_j^N \varphi^{\dagger}_j(x_j) \prod_\ell \varphi_2(x_\ell) |0\rangle.$$

is an eigenstate of $H_0 = -i \sum_j \partial_j$, for any choice of $\{p\}$,

but not yet with the correct boundary conditions!

What choice of p's corresponds to , or $p_1(p)$, $p_2(p)$.

A general eigenstate $|\psi\rangle$ of H_0, e.g.



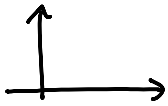
B.C. for in-state

Expand  $\Big\} = \sum_{\{p\}} A(\{p\}) \, |\{p\}\rangle$, with $\sum p = p_0$

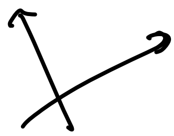
If this can be done, then you can calculate S-matrices. ...

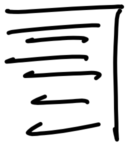
Here, we'll just worry about ground states only.

Fock



Good and Bethe want:





$\equiv$ is ground state of H_0, usually written in Fock basis, but it does not depend on basis.

$|\{p\}\rangle_0$ is Bethe basis of eigenstates of free Hamiltonian.

Auxiliary problem: find ground state of H_0, expressed in Bethe basis. and do this on a ring

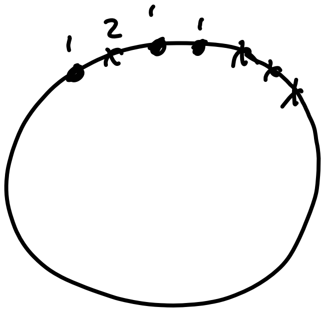
In general, we need to find a general mapping $|\varphi_n\rangle \longrightarrow |\psi_n\rangle$

Finite temperature $\quad \rho_0 = \sum_n p_n |\varphi_n\rangle\langle\varphi_n| \longrightarrow \rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$

Comment: $\mathcal{U}^\dagger(0,t_0)\,\rho_0\,\mathcal{U}(0,t_0)$ $\to \rho_s$ (out of eq., this does not work).

in Equil. $e^{-\beta H}$ N.A. + Benjamin Doyon

Back to more modest question: find $|gs\rangle$ in Bethe basis!



impose periodicity on wave functions:

$$e^{i p_i^{(1)} L} = e^{i \sum_\ell \Phi(p_i^1, p_\ell)}$$

$$e^{i p_i^{(2)} L} = e^{i \sum_\ell \Phi(p_n^{(2)}, p_\ell)}$$

$$i p_j^1 L = i \sum_\ell \Phi(p_j^1, p_\ell) + 2\pi I_j^{(1)}$$

$p^2 \ldots$

$\Big\} \Rightarrow$ get integral equations for $\rho^n(p)$

more generally:

$$E = \sum k_j + k_0 = \sum p_j$$

many combinations.

$$= \left(\prod_{k_j=-\infty}^{0} e^{i k_j x_j} \right) e^{i k_0 x}$$

$$= \sum A_{\{p\}} \, e^{i \sum p_j x_j + i \sum_{ij} \Phi(p_i, p_j)\, \mathrm{sign}(x_i - x_j)}$$

Calculate $A_{\{p\}}$, on finite ring, take $L \to \infty$ at the end.

Student: Sung-Po Chow calculates the $A\{p\}$