Lecture 4

BAEq:
$$N^\ell \theta(2\Lambda_\gamma - 2) + \theta(2\Lambda_\gamma) = -2\pi I_\gamma + \sum_{\delta=1}^{M} \theta(\Lambda_\gamma - \Lambda_\delta) \qquad (51.1)$$

$$\{I_\gamma\} \overset{\text{solve}}{\Longrightarrow} \{\Lambda_\gamma\}$$

$$E = \underset{\underset{\frac{N_\varepsilon}{L}}{\uparrow}}{D} \sum_\gamma \left[\theta(2\Lambda_\gamma - 2) - \pi\right] + \frac{2\pi}{L} \sum_j n_j \Big] = E\{I_\gamma, n_i\}$$

GS: $I_{min} \le I^0_\gamma \le I_{max} = \frac{N - M - 1}{2}$

Thermod. limit: $N, L \to \infty$

For $\{I^0_\gamma\}$: $\sigma_0(\Lambda) = f(\Lambda) - \int d\Lambda\, K(\Lambda - \Lambda')\, \sigma_0(\Lambda')$

$$f = \frac{2c}{\pi}\left[\frac{N^\ell}{c^2 + 4(\Lambda - 1)^2} + \frac{1}{c^2 + 4\Lambda^2}\right]$$

$$K = \frac{1}{\pi} \frac{c}{c^2 + \Lambda^2}$$

$$\tilde{\sigma}_0(p) = \tilde{f}(p) - \tilde{K}(p)\, \tilde{\sigma}_0(p) \Rightarrow \tilde{\sigma}_0(p) = \frac{\tilde{f}(p)}{1 + \tilde{K}(p)}.$$

N^e fixed :

$$\square \times \square = \square\square \quad 11 \, , \; \frac{12+21}{2} \, , \; 22$$

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \quad \begin{array}{c} 1 \\ 2 \end{array}$$

Young tableau : operator of permutations which acts on reducible product of irreducible representations of rotation group which breaks it up into a sum irreducible representations

YT block-diagonalizes

YT: rows : symmetric
columns: antisymmetric.
tells about symmetries of indices $A_{a_1 a_2}$ under interchange.

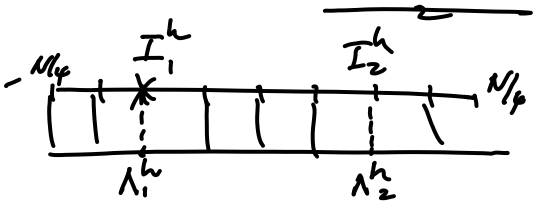
$\boxed{1}$ as operator is a symmetrizer: $\frac{1 + P_{12}}{2}$

N^e = fixed.

gs: $M = N/2$ singlet

exc : $M = \frac{N}{2} - 1$ triplet

$$I_{max} = \frac{N - (\frac{N}{2} - 1) - 1}{2} = N/4$$



$\Rightarrow$ # of slots $= 2\frac{N}{4} + 1 = \frac{N}{2} + 1$

of $I_\gamma = M = N/2 - 1 \Rightarrow$ two holes.

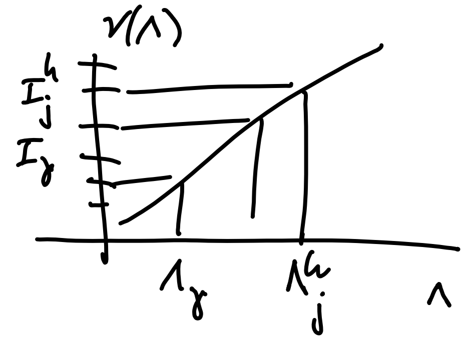
How to determine Λ_i^h ?

Choose $I_\gamma \longrightarrow$ determine Λ_γ

have been determined

$$\text{define} \quad \nu(\Lambda) = \frac{-1}{2\pi}\left[N^e \theta(2\Lambda - 2) + \theta(2\Lambda) - \sum_{\delta=1} \theta(\Lambda - \Lambda_\delta)\right]$$

$\uparrow$ counting function



when Λ hits Λ_γ, then $\nu(\Lambda) = I_\gamma$ (51.1)

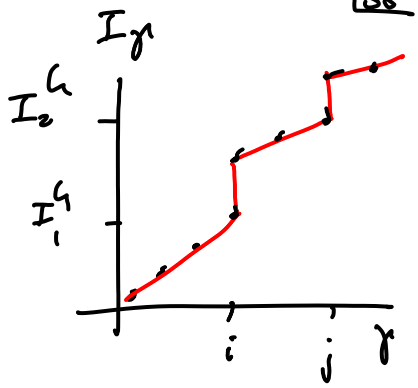
$\Rightarrow$ define Λ_j^h as the value where $\nu(\Lambda_j^h) = I_j^h$

Now:
$$\frac{d\nu(\Lambda)}{d\Lambda} = \sigma(\Lambda) + \sigma^h(\Lambda) = \text{density of states.}$$

# of slots = # of states = holes or not hole.

$$\sigma^h(\Lambda) = \delta(\Lambda-\Lambda_1^h) + \delta(\Lambda-\Lambda_2^h)$$

$$\uparrow \quad \frac{\partial\Theta(\Lambda)}{\partial\Lambda}$$



$$\sigma^h(\Lambda) + \sigma(\Lambda) = f(\Lambda) - \int d\Lambda' \, K(\Lambda-\Lambda')\,\sigma(\Lambda')$$

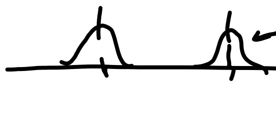
FT: $\updownarrow$

$$e^{-ip\Lambda_1^h} + e^{-ip\Lambda_2^h} + \tilde\sigma(p) = \tilde f(p) - \tilde K(p)\,\tilde\sigma(p)$$

$$\tilde\sigma(p) = \frac{\tilde f(p) - e^{-ip\Lambda_1^h} - e^{-ip\Lambda_2^h}}{1+\tilde K(p)} = \tilde\sigma_0(p) + \Delta\tilde\sigma(p)$$

backflow of other states when you take out 2.

$$\Delta\tilde\sigma(p) = -\frac{e^{ip_1\Lambda_1} + e^{ip_2\Lambda_2}}{1+\tilde K(p)}$$



← width of δ-f. in Λ-space

$$\tilde K(p) = e^{-c|p|}$$

$$\hookrightarrow \quad \tilde\Delta(p) = -\frac{e^{\frac{c}{2}|p|}}{2\,\mathrm{ch}\,\frac{c}{2}p}\left(e^{-ip\Lambda_1^h} + e^{-ip\Lambda_2^h}\right)$$

$$\Delta E^t = \int d\Lambda \; \Delta\sigma(\Lambda)\,\left[\Theta(2\Lambda-2) - \pi\right]$$

$$\boxed{\Delta E^t = 2D\left[\tan^{-1} e^{\frac{\pi}{2}(\Lambda_1^h-1)} + \tan^{-1} e^{\frac{\pi}{2}(\Lambda_2^h-1)}\right]}$$

= positive!, always ⇒ $M = N/2$ is ground state.

Field theory & renormalization:

$$T,\ h,\ p \ll \begin{cases} D \to \infty \\ J_0 \rightsquigarrow\ ? \end{cases}$$ how should J change to keep physics fixed?

$$\Delta E^d = 2D\,\underbrace{\tan^{-1}\left(e^{\frac{\pi}{c}(\Lambda^h - 1)}\right)}_{\sim\, e^{\frac{\pi}{c}(\Lambda^h-1)} \text{ if arg} \to 0.}$$ should not change of $D \to \infty$

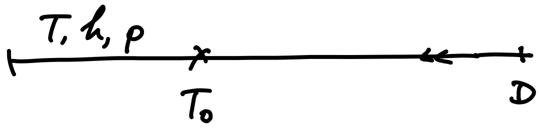
$$\frac{\pi}{c}\Lambda = X$$

so: $D \to \infty$, means $c \to 0$: why?

$$\underbrace{2D\, e^{-\pi/c}}_{\equiv T_0}\, e^{X} = 2T_0\, e^{X}$$

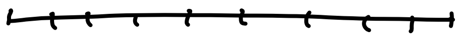
so, keep T_0 fixed !!

Claim: this works for all other excitations, too.

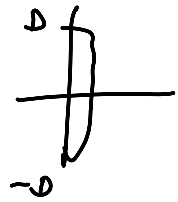


Question: Scaling arguments show T_0 is more complicated

RG defined conventionally means: either: w/ cutoff: use lattice,



or bandwidth



$\Rightarrow$ gives: $$T_0 = D\, e^{\left(-\frac{\pi}{2J} + \frac{1}{2}\ln J + \dots J + \dots\right)}$$

it was claimed that first two terms are universal independent of RG scheme.

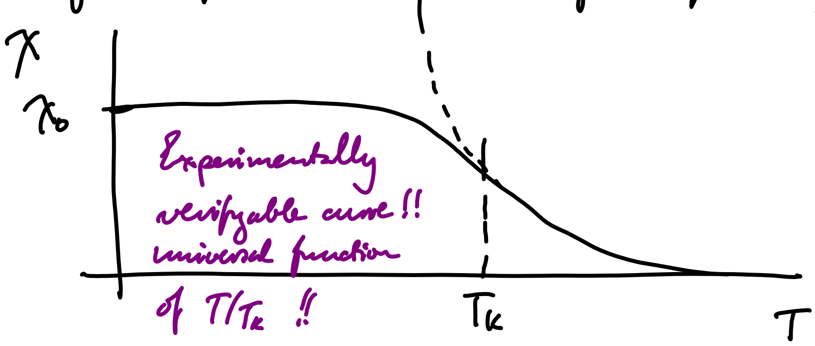
But, Bethe-Ansatz gives $\tilde{T}_0 = \mathcal{D}\, e^{-\frac{\pi}{2J_{\mathcal{D}}}}$

But this assumed: decimation is done sufficiently slowly, that bare quantities and renormalized quantities are not analytically related.

But Bethe Ansatz bare parameters $\mathcal{D}$ and $J_{\mathcal{D}}$ are not analytically related to bare parameters of other cutoff schemes.

So, identify $\tilde{T}_0 \equiv T_0$.

Normalization of T_0 : need physical quantity



1-loop

$$\chi^i = \frac{\mu^2}{T}\left[1 - \frac{1}{\ln T/T_K^c} + \frac{\frac{1}{2}\ln\ln T/T_K}{\ln^2 T/T_K} + \frac{\tilde{c}}{\ln^2 T/T_K} + \dots\right]$$

Convention: normalize T_K such that $c = 0$. $\;\; \approx \tilde{c} = 0 \Rightarrow c = 1$

why:
$$\frac{1}{\ln T/T_K - \ln c} = \frac{1}{\ln T/T_K} + \frac{\ln c}{\ln^2 T/T_K} + \dots$$

T_K is the value defined such that χ has no $\frac{?}{\ln^2 T/T_K}$

Also, $\quad \chi_0 = \dfrac{\mu^2}{\pi T_0} \quad$ is

Wilson number: $\quad \underset{4\pi}{W} \equiv \dfrac{T_K}{4\pi T_0} = e^{(C - 1/4)}$ $\quad$ Euler's #

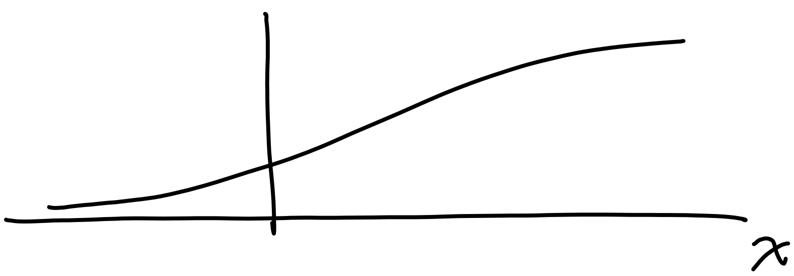
$\xrightarrow{LA}$ 0.102676

$\xrightarrow{\text{Wilson}}$ 0.1032 ± 0.0005

Lowenstein
Andrei

↑ universal number
(relates strong-coupling to weak-coupling physics)

ΔE^t :



$$\Delta E^t = 2T_0 e^{\chi}$$

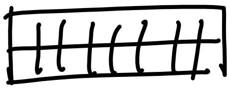
[in field theory, if you had left-movers, we would get $\Delta E^t = 2T_0\left[e^{\chi} + e^{-\chi}\right]$
$= \dfrac{T_0}{\cosh\chi} \Rightarrow$ gap!!]

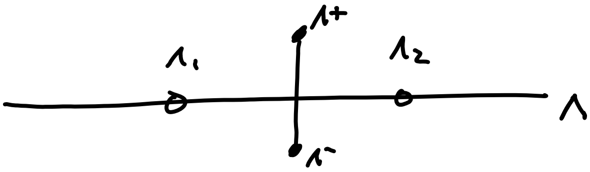
Summary: triplet : $S = 1$ state consists of

$$\Delta E = 2T_0 e^{\chi_1} + 2T_0 e^{\chi_2}$$

tempting to say that each hole carries spin 1/2. $\quad$: $1/2 + 1/2 = 1$.

To substantiate this: see whether you can make excited triplet out of this!





2-string

position first found by Lowenstein & Andrei

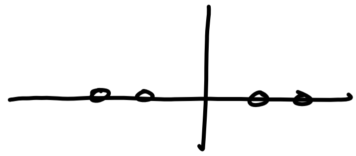
BA Equations have complex solutions, called "strings":

$$\Lambda^{\pm} = \frac{\Lambda_1 + \Lambda_2}{2} \pm i\, c/2$$

Energy: still $\Delta E^t = 2T_0 \left(e^{\chi_1} + 2T_0^{\chi_2} \right)$ $\qquad \frac{\pi}{c}\Lambda = \chi$

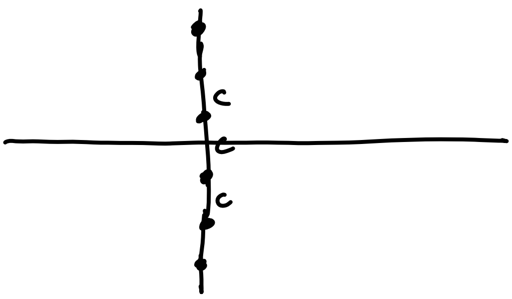
= degenerate with triplet in therm. limit (but there are $1/N$ corrections).

other examples:



Takahashi: string hypothesis (not fully precise): complex solutions of BAE take form of n-strings

n-string: symmetrically arranged Λ's:



$I_\gamma \to I_\gamma^{(1)} \to \sigma_{(1)}(\Lambda)$ for 1-strings

$I_\gamma \to I_\gamma^{(2)} \to \sigma_{(2)}(\Lambda)$ for 2-strings.

general solution characterized by $I_\gamma^{(n)} \rightsquigarrow$ specifies the n-strings.

$\rightsquigarrow \sigma_n(\Lambda)$, $\sigma_n^h(\Lambda)$, etc...

$$\sigma_1(\Lambda) + \sigma_1^h(\Lambda) = \int d\Lambda' K \,\sigma_1(\Lambda') + f \qquad \text{for real } \Lambda\text{'s.}$$

in general, more complicated σ's enter here leading to coupled equations for σ_n, σ_n^h,

String hypothesis: general solution described by densities of n-strings.

BA for all strings:

$$\sigma_n^h(\Lambda) + \sum_{m=1}^{\infty} A_{nm}\, \sigma_m(\Lambda) = f_n(\Lambda)$$

typically, you solve this explicitly only for few excitations.
For thermodynamics, you don't solve it explicitly!

Thermodynamics:

Partition f: $Z = \mathrm{Tr}\, e^{-\beta E} = \sum_{\{n_j, I_\gamma\}} e^{-\beta E\{n_j, I_\gamma\}}$

Add mag. field: $H = H + h(\underbrace{S^z + \int dx\, \psi^\dagger \sigma^z \psi}_{S^z_{tot}})$ $\qquad [S^z_{tot}, H] = 0$

↑ diagonalized by Eigenstates of H.

brothers

C.N. Yang + C.P. Yang: thermodynamic Bethe Ansatz (TBA)

YY

(studied bosons interacting with local potential; generalized to SU(2) by Gaudin, Takahashi)

to Kondo: Lowenstein + Andrei; Wiegmann + Fadeev

YY observed: in practice, $E = E\{\sigma_n, \sigma_n^h\}$ (only densities matter!)

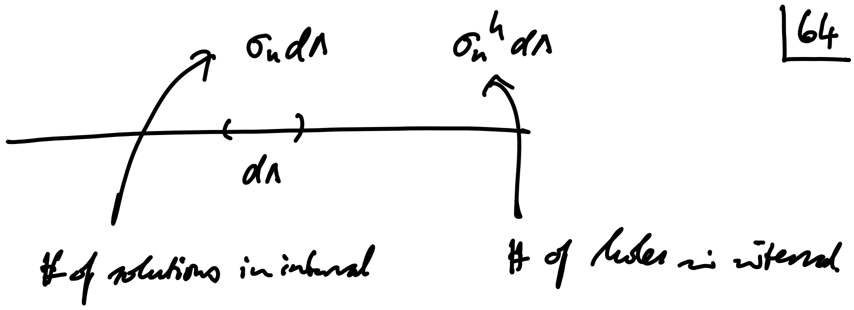
$$E = \int \mathcal{D}\lambda \int \mathcal{D}\sigma(n_j) \int \mathcal{D}\sigma_n\, \mathcal{D}\sigma_n^h\, e^{-\beta E\{\sigma_n, \sigma_n^h, \sigma_{n_j}\} - T S\{\sigma_n, \sigma_n^h, \sigma_{n_j}\} + \lambda(\mathrm{BAE})}$$

counting factor

given $\{I_\gamma\} \Rightarrow \sigma_n, \sigma_n^h$ different $\{I\}$'s can give same σ_n, σ_n^h

$\{I_\gamma'\} \nearrow$ microscopic macroscopic

YY: consider interval $d\Lambda$



total number of slots is $(\sigma_n + \sigma_n^h)\, d\Lambda$

# of choices that can be made is $\dfrac{\left((\sigma_n + \sigma_n^h) d\Lambda\right)!}{(\sigma_n d\Lambda)!\,(\sigma_n^h d\Lambda)!}$ $\qquad$ how many $\{I_n, I_n^\gamma\}$ contribute

entropy: $$dS[\sigma_n, \sigma_n^h] = \ln \frac{\left((\sigma_n + \sigma_n^h) d\Lambda\right)!}{(\sigma_n d\Lambda)!\,(\sigma_n^h d\Lambda)!}$$

$$S = \int dS = \int d\Lambda \frac{dS}{d\Lambda}$$

subject to constraint that BAE are satisfied

Saddle-Point (in thermodynamic limit, only one configuration survives, namely saddle point s.). YY

Everything is expressed in terms of functions $\eta_n(\Lambda) = \dfrac{\sigma_n^h(\Lambda)}{\sigma_n(\Lambda)}$, $n = 1, \dots \infty$

2S in general !!

$$\Rightarrow \text{free energy} \quad \underset{S=1/2}{F^i} = -\frac{T}{2\pi}\int dx \; \frac{\ln\left(1 + \eta_1(x, h/T)\right)}{\operatorname{ch}\left(x - \ln T_0/T\right)}$$

where η_n satisfy TBA equations: $\;G$ is integral operator: $\;G f(x) = \displaystyle\int \frac{dx'}{\operatorname{ch}(x-x')} f(x')$

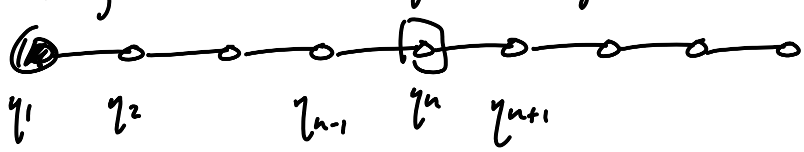
driving term

$$\ln \eta_1 = -e^{x} + G \ln(1+\eta_2)$$
$$\vdots$$
$$\ln \eta_n = G \ln(1+\eta_{n-1}) + G \ln(1+\eta_{n+1}),$$

where $\ln \eta_n \xrightarrow{n\to\infty} \dfrac{nh}{T}$, h enters as an asymptotic condition.

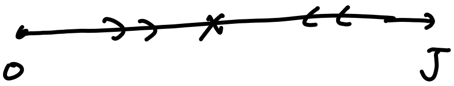
TBA converge quickly on computer.

Shorthand for these eq: driving term e^x $\quad \eta_{2S}$ is all you need !!



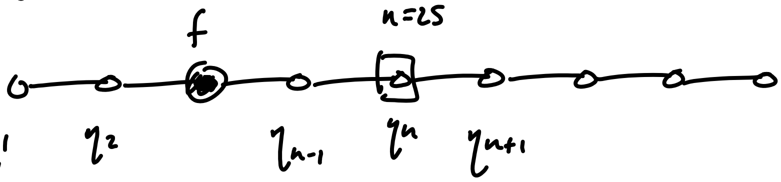
Claim: (Destri, Andrei; Tsvelik, Wiegmann, 1984)

For multichannel Kondo: Ψ_{am} : Nozière & Blandin $\quad$ Non-fermi liquid!

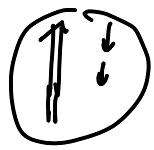


$m = 1, \dots, f \qquad f = \#$ of flavors

Only change: driving term enters at level f!



NB: if $\frac{f}{2} \le s \Rightarrow$ underscreened



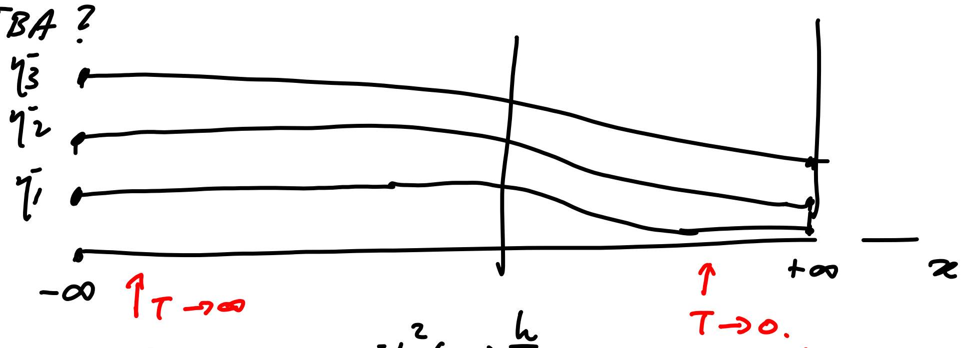
$s' = s - \frac{1}{2}f$ = partially screened

$\frac{f}{2} > s \Rightarrow$ overscreened

simplest case: 2 channels, $s = 1/2$

How does this emerge from TBA?

1CK: $f=1$



$$\eta_1 \sim e^{-e^x + g h} \; ; \qquad \eta_1^- = \frac{\sinh^2 (n+1)\frac{h}{T}}{\sinh^2 h/T} - 1$$

$$\eta_1^+ = \frac{sh^2 n\, h/T}{sh^2 h/T} - 1$$

where does $\dfrac{1}{ch\left(x - \ln \frac{T_0}{T}\right)}$ contribute? in F.

when $\eta \to$ const, $g \to \delta$-function, $\to g = 1/2$

$$F^i_{s=1/2} = -\frac{T}{2\pi} \int dx \, \frac{\ln(1 + \eta_1(x, h/T))}{ch(x - \ln T_0/T)}$$

becomes constant for $T \to \infty$ or 0

Brillouin function.

$$F^i_{1/2}(T\to\infty) = -\frac{T}{\cancel{2\pi}} \, \frac{1}{2} \ln \overset{(\eta_1^-)}{\frac{sh^2 (2S+1) h/T}{sh^2 (h/T)}} = -\frac{T}{2\pi} \ln \frac{sh(2S+1) h/T}{sh(h/T)} = \left(\begin{array}{c}\text{free spin}\\ S=1/2\end{array}\right)$$

$S=1/2$: $\to 2\cosh h/T$

Z of $s=1/2$ in mag field: $$Z = e^{h/T} + e^{-h/T} = 2\cosh h/T = e^{-\beta F}$$

$$\Rightarrow F = -T \ln (2\cosh h\mu/T)$$

$$F^i_{1/2}(T\to 0) = -T \, \frac{1}{2} \ln \frac{sh(2S) h/T}{sh(h/T)} = \text{free energy of } S - 1/2.$$

So, we see crossover from spin S to $S - 1/2$ $\Rightarrow$ screening!!

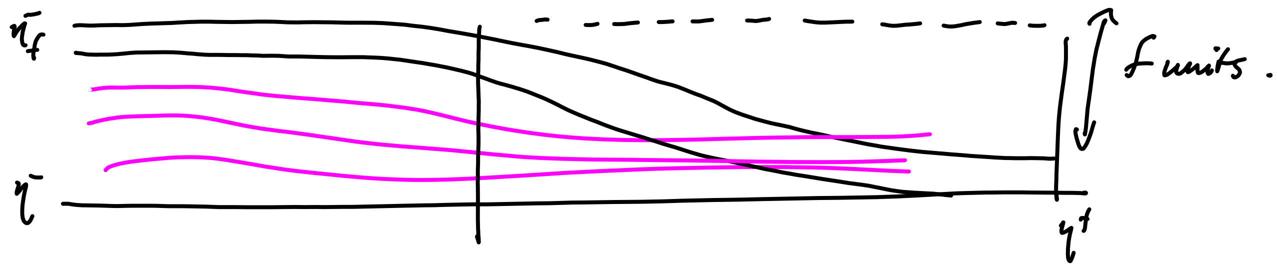
$s = 1/2 \longrightarrow s = 0.$

$$\chi \xrightarrow{T\to\infty} \frac{\mu^2}{T}\left[1 - \frac{1}{\ln T/T_K} + \frac{1}{2}\frac{\ln\ln}{\ln^2} + \dots\right]$$

$\chi \xrightarrow{T=0}$ also known. ✓

if $f=1$: η goes down by 1 unit

Multichannel : $f \neq 1$: η_f is driven to 0 !



asymptotic behavior for $x \to -\infty$, $T \to \infty$ is the same.

$h=0$ limit.

$$\bar\eta_n = \frac{\mathrm{sh}^2 (n+1)\frac{h}{T}}{\mathrm{sh}^2 h/T} - 1$$

$$\eta^f_{n\geq f} = \frac{\mathrm{sh}^2 (n+1-f)\frac{h}{T}}{\mathrm{sh}^2 h/T} - 1$$

$$\eta^f_{n<f} = \frac{\sin^2 (n+1)\left(\frac{\pi}{f+2}\right)}{\sin^2 \frac{\pi}{f+2}} - 1$$

if $s \geq \frac{1}{2}f$: $s \to s - \frac{1}{2}f$ underscreened !!

$s = \frac{1}{2}f$: $s \to 0$ screened.

if $s < \frac{1}{2}f$:

underscreened: $F^i \xrightarrow[h=0]{T\to 0} -T \ln \left(\frac{\sin(2s+1)\frac{\pi}{f+2}}{\sin\frac{\pi}{f+2}} \right)$

Entropy: $S = \frac{dF^i}{dT} = \ln \left(\frac{\sin(2s+1)\frac{\pi}{f+2}}{\sin\frac{\pi}{f+2}} \right).$

$s=1/2$: usually: at $T=0$ $\quad S(s=1/2) = \ln(2)$

here: $\boxed{S(s=1/2) = \ln\sqrt{2}}$!!

Interpretation in RG: "intermediate coupling with strange properties"

Physical interpretation: consider finite size:

\# of states close to E_0 is $\sim L^{\nu}$ as $L \to \infty$

usually: $\nu = 1$; here $\nu \neq 1$.

specific heat: $C_V = T^{\frac{4}{f+2}-1}$

$\chi = T^{\frac{4}{f+2}}$

for Fermi liquid: these powers are non-fractional!

High-T limits can be obtained perturbatively.

low-T limit: non-perturbative !!