Lecture 3 QIS – Problem

$$(Z_j)^{b_1 \dots b_N}_{a_1 \dots a_N} = \left(S^{jj-1} \dots S^{j1} \, S^{jN} \dots S^{jj+1} \right)^{b_1 \dots b_N}_{a_1 \dots b_N}$$

$$S^{ij} = \frac{(\alpha_i - \alpha_j) I^{ij} + ic P^{ij}}{(\alpha_i - \alpha_j) + ic}$$

acts on space of N spins.

$$(I^{ij})^{b_i b_j}_{a_i b_i} = \delta^{b_i}_{a_i} \delta^{b_j}_{a_j} = \uparrow \uparrow$$

$$(P^{ij}) = \delta^{b_j}_{a_i} \delta^{b_i}_{a_j} =$$ 

$$S^{ij} \, (S^{ik})^{a_i' a_k''}_{a_i a_k} (S^{jk})^{a_j' a_k'}_{a_j a_k''} \qquad A_{a_i' a_j' a_k'}$$

$$(S^{ij})^{a_i' a_j' a_k'}_{a_i a_j a_k} = (S^{ij})^{a_i' a_j'}_{a_i a_j} \, \delta^{a_k'}_{a_k}$$

$$(S)^{a_i' \dots a_N}_{a_i \dots a_N} = \delta \dots (S^{ij})^{a_i' a_j'}_{a_i a_j} \dots \delta \dots$$

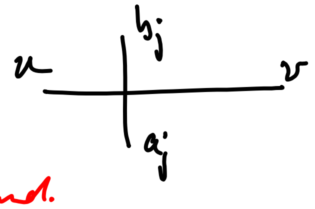
Define: $$S^{ij}(\alpha) = \frac{\alpha I^{ij} + ic P^{ij}}{\alpha + ic}$$

Continuous YB holds: $$S^{ij}(\alpha) \, S^{ik}(\alpha+\beta) \, S^{jk}(\beta) = S^{jk}(\beta) \, S^{ik}(\alpha+\beta) \, S^{ij}(\alpha)$$

α = spectral parameter.

Z acts on V^N

(V is space of 1 spin) add $V^N \otimes V$

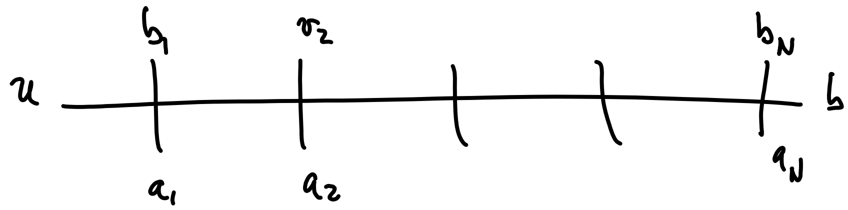


Define: $$S^{jA}(\alpha) = \frac{\alpha + ic\, P^{jA}}{\alpha + ic}$$

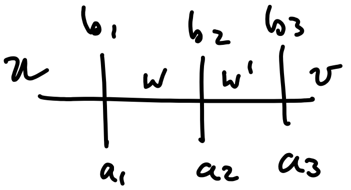
flip order around.

Monodromy Matrix: $$\Xi(\alpha) = S^{1A}(\alpha-\alpha_1)\, S^{2A}(\alpha-\alpha_2) \cdots S^{NA}(\alpha-\alpha_N)$$

($\alpha_1 \dots \alpha_N$ are arbitrary numbers.)



Explicitly:



$$= \sum_w S^{a_2 w}_{a_1 u}\; S^{b_2 w'}_{a_2 w}\; S^{b_3 v}_{a_3 w'}$$

"quantum index" auxiliary

$$u \overset{b_1}{\underset{a_1}{S}} w - w \overset{b_2}{\underset{a_2}{S}} w' - w' \overset{b_3}{\underset{a_3}{S}} v$$

Theorem (Belavin):

$$Z_j = \underbrace{\mathrm{Tr}_A}_{\text{aux space}}\; \Xi(\alpha = \alpha_j)$$

$S^{jA}(\alpha = \alpha_j) = P^{jA}$

$$\Xi(\alpha_j) = S^{1A}(\alpha-\alpha_1)\, S^{2A}(\alpha-\alpha_2) \cdots \underbrace{P^{jA}\, S^{j+1\,A}(\alpha-\alpha_{j+1})\, P^{jA}}_{S^{j+1\,j}}\, P^{jA} \quad S^{NA}(\alpha-\alpha_N)$$

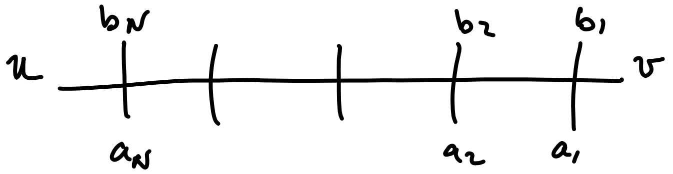
in this way, pull P^{jA} to the end.

$\mathrm{Tr}_A\, P^{jA} = \frac{1}{2}$, so you recover Z.

take 3: $\alpha = \alpha_2$ $$\mathrm{Tr}_A\; S^{3A}(\alpha_2 - \alpha_3)\, \underbrace{S^{2A}(\alpha_2 - \alpha_2)}_{P^{2A}}\, S^{1A}(\alpha_2 - \alpha_1)\; P^{2A}\; P^{2A}$$

$$= \mathrm{Tr}_A\; P^{2A}\, S^{3A}(\alpha_2-\alpha_3)\, \cancel{P^{2A}}\, P^{2A}\, S^{21}(\alpha_2-\alpha_1)\, \cancel{P^{2A}}$$

$$= \mathrm{Tr}_A\; S^{32}(\alpha_2-\alpha_3)\, P^{2A}\, S^{21}(\alpha_2-\alpha_1) = \underbrace{S^{32}(\alpha_2-\alpha_3)\, S^{21}(\alpha_2-\alpha_1)}_{=Z_2}$$



$\Xi(\alpha) \rightsquigarrow$ gives all the $Z_j = \mathrm{Tr}_A\, \Xi(\alpha_j)$

Recall: $[Z_j, Z_\ell] = 0$; we'll prove this by proving

$$[\,\mathrm{Tr}_A\, \Xi(\alpha)\,,\ \mathrm{Tr}_A\, \Xi(\beta)\,] = 0$$

then $Z_j = A(\alpha_j) + B(\alpha_j)$

$A : uv = \uparrow\uparrow$
$B : \quad\ = \uparrow\downarrow$
$C : \quad\ = \downarrow\uparrow$
$D : uv = \downarrow\downarrow$

Consider $\quad \Xi(\alpha)^{b_1 \ldots b_N,\, u}_{a_1 \ldots a_N,\, v} = \begin{pmatrix} A^{b_1\ldots b_N}_{a_1 \ldots a_N} & B^{\cdots}_{\cdots} \\ C^{\cdots}_{\cdots} & D^{\cdots}_{\cdots} \end{pmatrix}$

Tensor product: $\quad \Xi(\alpha) \otimes \Xi(\alpha') = 4\times 4$ matrix

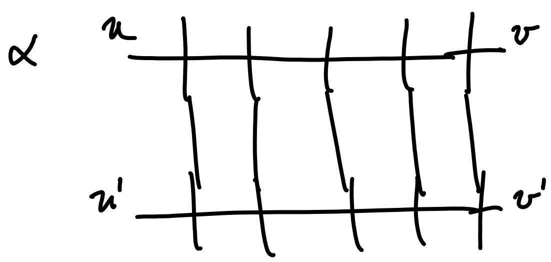
$$= \begin{pmatrix} A\,A' & A\,B' & B\,A' & B\,B' \\ A\,C' & A\,D' & B\,C' & B\,D' \\ C\,A' & C\,B' & D\,A' & D\,B' \\ C\,C' & C\,D' & D\,C' & D\,D' \end{pmatrix}$$

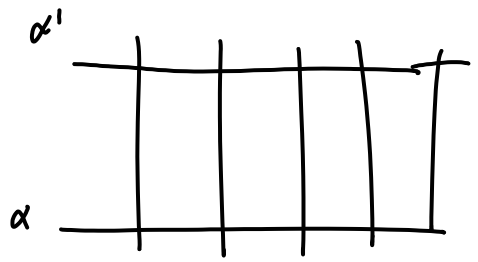
quantum indices are contracted

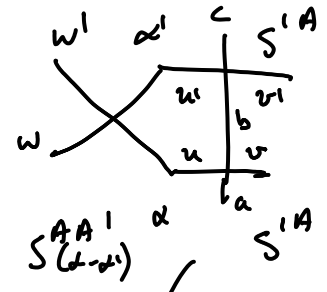
$$\Xi(\alpha')\,\Xi(\alpha) = \begin{pmatrix} A'A & A'B & B'A & B'C \\ A'C & A'D & B'B & B'D \\ C'A & C'B & D'A & D'B \\ C'C & C'D & D'C & D'D \end{pmatrix}$$

Claim: there exists a 4×4 matrix R, such that

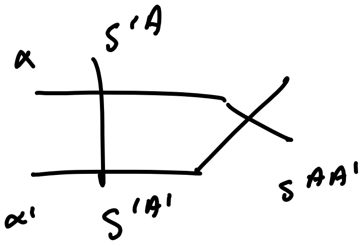
$$R(\alpha-\alpha')\,\Xi(\alpha)\,\Xi(\alpha') = \Xi(\alpha')\otimes\Xi(\alpha)\,R(\alpha-\alpha')$$







$=$

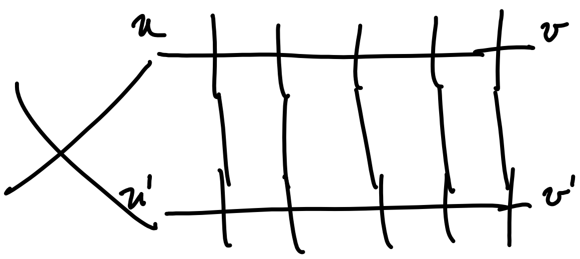


$$\overset{\shortparallel}{R}$$

$$S^{AA'}\, S^{'A}\, S^{'A'} = S^{'A}\, S^{'A}\, S^{AA'}$$

$$\left(S^{AA'}_{(\alpha-\alpha')}\right)^{u'w'}_{uw} \left(S^{'A}_{(\alpha)}\right)^{bu}_{av} \left(S^{'A'}(\alpha')\right)^{cv'}_{bu'}$$

auxiliary space is tool to keep track of "hidden indices" in YBEq.

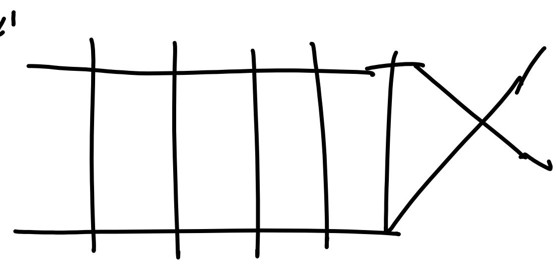


$R\ \Xi \otimes \Xi'$

$=$

holds if $R=S$

$=$



$\Xi' \otimes \Xi\ R$

$$R\ \Xi \otimes \Xi' = \Xi' \otimes \Xi\ R$$

So, we end up with 16 operator relations:

$$A(\alpha)\, B(\alpha') = u(\alpha'-\alpha)\, B(\alpha')\, A(\alpha) + v(\alpha'-\alpha)\, B(\alpha) A(\alpha')$$

$$D(\alpha)\, B(\alpha') = u(\alpha-\alpha')\, B(\alpha')\, D(\alpha) + v(\alpha-\alpha')\, B(\alpha) D(\alpha')$$

$$B(\alpha)\, B(\alpha') = B(\alpha')\, B(\alpha), \qquad u(\alpha) = \frac{\alpha + ic}{\alpha}$$

$$A(\alpha')\, A(\alpha) = A(\alpha)\, A(\alpha'), \qquad v(\alpha) = -\frac{ic}{\alpha}$$

Take Tr: $\quad \mathrm{Tr}_A\ R\ \Xi \otimes \Xi' R^{-1} = \Xi' \otimes \Xi$

$$= \Xi \otimes \Xi' = \Xi' \otimes \Xi$$

$$\Rightarrow \quad z\, z' = z'\, z$$

We want to diagonalize $Z(\alpha) = A(\alpha) + D(\alpha)$

we'll do this using B, C.

Step 1: $|\uparrow \ldots \uparrow\rangle = |\mathcal{F}\rangle$ is an eigenstate.

Step 2: $B(\lambda)\,|\mathcal{F}\rangle$ is an eigenstate, for certain λ

Step 3: $B(\lambda_1) \cdots B(\lambda_M)\,|\mathcal{F}\rangle$ " " , if $\lambda_1 \ldots \lambda_M$ satisfy BA equations

to show this, act with $(A+D)$ on $B\,|\mathcal{F}\rangle$, and move $A+D$ to right, using the algebra!!

$$S^{jA}(\alpha) = a(\alpha)\,1 + b(\alpha)\,P^{jA}$$

$$= a(\alpha)\,1 + b(\alpha)\,\tfrac{1}{2}\left(1 + \vec{\sigma}_j \cdot \vec{\sigma}_A\right)$$

$$= \left[a(\alpha) + \tfrac{1}{2}b(\alpha)\right] 1_j\, 1_A + \tfrac{b}{2}\,\vec{\sigma}_j \cdot \vec{\sigma}_A$$

$$= \begin{pmatrix} (a+\frac{b}{2})(\alpha)\,1_j & \frac{b}{2}\,\sigma_j^- \\ \frac{b}{2}\,\sigma_j^+ & (a-\frac{b}{2})(\alpha)\,1_j \end{pmatrix}$$

$$\Xi = \prod_j S^{jA}(\alpha) = \prod_j \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

Eg: $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $H = \begin{pmatrix} (a+\frac{b}{2})(\alpha-\alpha_1)\,1_{(1)} & b(\alpha-\alpha')\,\sigma^-_{(1)} \\ b(\alpha-\alpha_1)\,\sigma^+_{(1)} & (a-\frac{b}{2})\,1_{(1)} \end{pmatrix} \begin{pmatrix} (a+\frac{b}{2})(\alpha-\alpha')\,1_{(2)} & b(\alpha-\alpha_2)\,\sigma^-_{(2)} \\ b(\alpha-\alpha_2)\,\sigma^+_{(2)} & (\alpha-\frac{b}{2})\,1_2 \end{pmatrix}$

Act with this on: $|\uparrow\rangle_1 |\uparrow_2\rangle$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} |\uparrow\uparrow\rangle = \begin{pmatrix} (a+\frac{b}{2})(\alpha-\alpha_1)\,1_{(1)} & b(\alpha-\alpha')\,\sigma^-_{(1)} \\ 0 & (a-\frac{b}{2})\,1_{(1)} \end{pmatrix} \begin{pmatrix} (a+\frac{b}{2})(\alpha-\alpha')\,1_{(2)} & b(\alpha-\alpha_2)\,\sigma^-_{(2)} \\ 0 & (\alpha-\frac{b}{2})\,1_2 \end{pmatrix} |\uparrow\rangle_1 |\uparrow\rangle_2$$

multiply out, take trace: $a + b/2 = 1$, since S-matrix is unitary

$A(\alpha)\,|F\rangle = |F\rangle \quad \Longrightarrow$ step 1 is now complete.

$D(\alpha)\,|F\rangle = \Delta(\alpha)\,|F\rangle$

with $\displaystyle \Delta(\alpha) = \prod_{j=1}^{N} \frac{\alpha-\alpha_j}{\alpha-\alpha_j+ic}$

Step 2: Read off B: $B = 1_{(1)}\,\sigma^-_{(2)} + \sigma^-_{(1)}\,1_2$

So: $B = \sum_j \sigma_j^- \times$ stuff $\leftarrow 1, \sigma^z$'s, etc.

Consider now: $\underbrace{(A(\alpha)+D(\alpha))}_{z} B(\lambda_1) \;\ldots\; B(\lambda_N)\,|F\rangle$ unwanted terms

see p. 29. commute to the right, pick up V-terms:

Set condition that extra terms vanish: $= z(\alpha_i\, \lambda_1 \ldots \lambda_M)\, B(\lambda_1) \ldots B(\lambda_N)\,|F\rangle$

Eigenvalue: $z(\alpha;\lambda_1,\dots,\lambda_M) = \prod_{\gamma=1}^{M} u(\lambda_\gamma-\alpha) + \Delta(\alpha)\prod u(\alpha-\lambda_\gamma)$

Provided that
$$\prod_{\delta=1}^{M} \frac{\lambda_\delta-\lambda_\gamma+ic}{\lambda_\delta-\lambda_\gamma-ic} = \prod_{i=1}^{N} \frac{\lambda_\gamma-\alpha_i}{\lambda_\gamma-\alpha_i+ic}$$

These equations fix the λ's needed such that $B\dots B|F\rangle$ is an eigenstate.

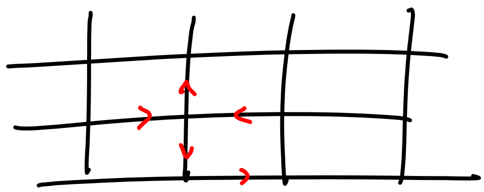
define $\lambda_\delta = \Lambda_\delta - ic/2$

$$e^{ik_jL} = z(\alpha;\lambda_1\dots\lambda_M) = \prod_\gamma \frac{\Lambda_\gamma-\alpha_j-ic/2}{\Lambda_\gamma-\alpha_j+ic/2}$$

$$\prod_{\delta=1}^{M} \frac{\Lambda_\delta-\Lambda_\gamma+ic}{\Lambda_\delta-\Lambda_\gamma-ic} = \prod_{i=1}^{N} \frac{\Lambda_\gamma-\alpha_i-ic/2}{\Lambda_\gamma-\alpha_i+ic/2}$$

Historical comments:

Baxter: 6 or 8-vertex model

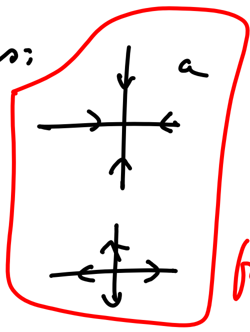


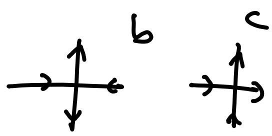
In principle there are 16 types of vertices

Restrictions: same number of in-and-out arrows;

8 possibilities:

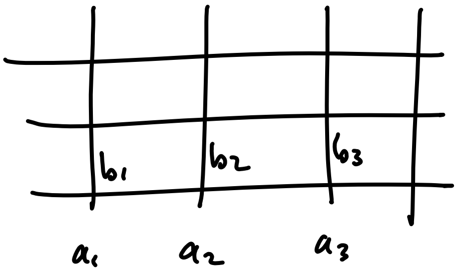
$$Z = \sum_{\text{configurations}} e^{\epsilon}$$

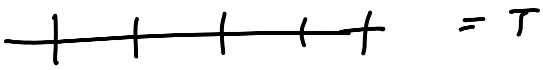




forbidden in 6-vertex model.

first solved by Lieb.

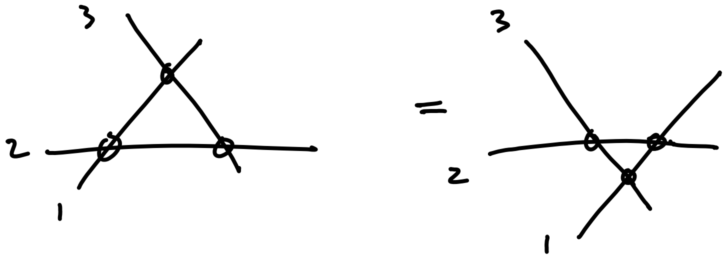




multiplying them gives partition function: $Z = \mathrm{Tr}\, T^M$, find eigenvalues $\lambda_1^M + \lambda_2^M$

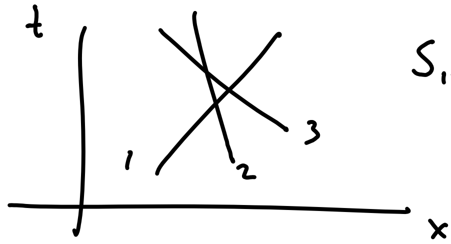
Baxter generalized Lieb to 8 vertices.

Baxter realized



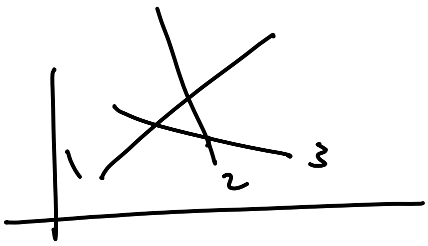
(sum over internal indices)

consider space-time picture of interacting particles



$S_{123} \overset{?}{=} S^{23} S^{13} S^{12}$

can it be factorized



$S^{23} S^{13} S^{23}$

factorization would make sense only if you can change the order without changing slope = momentum

Then Baxter introduced concept of commuting transfer matrices (spectral parameters) : $[Z(\alpha), Z(\beta)]$.

Monodromy matrix: introduced by Sklyanin

need to find coefficients

Yang diagonalized Z_j by direct Bethe Ansatz for $F \uparrow \ldots \downarrow \uparrow |F\rangle$