E_{10} and gravitational duality

Marc Henneaux

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It is thought to be relevant to the so-called problem of "hidden symmetries".

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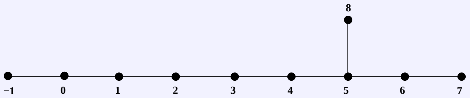

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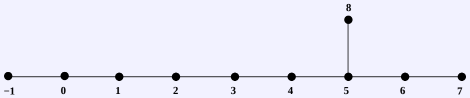

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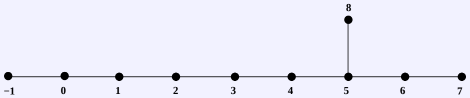

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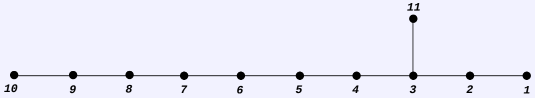

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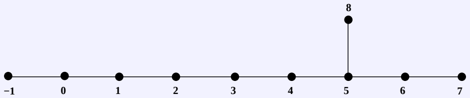

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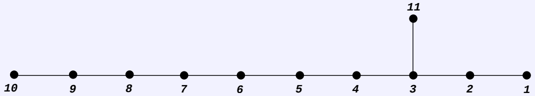

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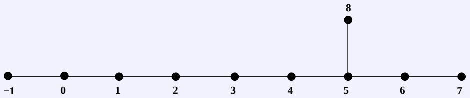

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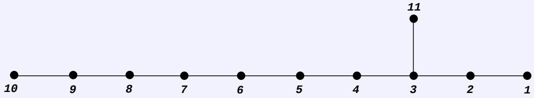

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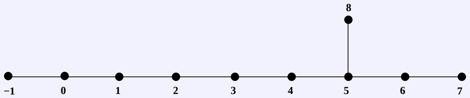

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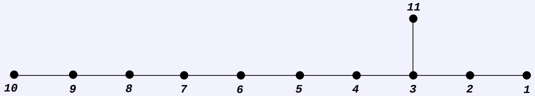

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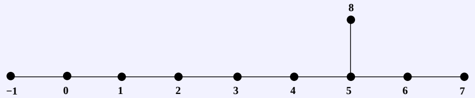

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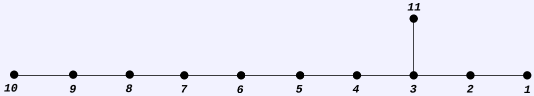

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This might be due to an insufficient understanding of duality.

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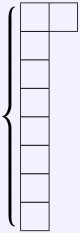

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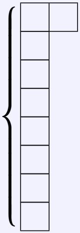

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If E_{10} is a symmetry of the action, what form should we expect the action to take ?

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- finally conclude and mention some open questions.

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Old result, Deser-Teitelboim 1976 - For more recent considerations, Deser-Gomberoff-Henneaux-Teitelboim 1997

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Another answer is no, because the action $S[q(t)]$ is not invariant under $q \to q' = \cos\alpha\, q - \sin\alpha\, \dot{q}$ and $\dot{q} \to \dot{q}' = \sin\alpha\, q + \cos\alpha\, \dot{q}$.

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The second answer is nonsense because there is no transformation of $q(t)$ (the dynamical Lagrangian variable !) that gives $q \to q' = \cos\alpha\, q - \sin\alpha\, \dot{q}$ and $\dot{q} \to \dot{q}' = \sin\alpha\, q + \cos\alpha\, \dot{q}$ (for general $q(t)$'s), since $q \to q' = \cos\alpha\, q - \sin\alpha\, \dot{q}$ implies $\dot{q} \to \dot{q}' = \cos\alpha\, \dot{q} - \sin\alpha\, \ddot{q}$.

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So the question is : is there a transformation of the dynamical variable $q(t)$ such that (i) it coincides on-shell with the given transformation ; and (ii) it leaves the action invariant.

The answer is affirmative : it is just time translation, $q(t) \to q'(t) = q(t - \alpha)$, which is indeed a symmetry transformation that takes the required form on-shell.

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But one encounters the following problem !

Theorem : There is no variation of A_μ that yields the above duality transformations of the field strengths.

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Once A_μ is introduced, $dF = 0$ is an identity. But $dF' = 0$ does not hold off-shell (unless α is a multiple of π but then F and its dual are not mixed), where F' is the new field strength $\cos\alpha\, F - \sin\alpha\, {}^*F$ after duality rotation. Hence there is no A' such that $F' = dA'$.

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It follows from this theorem that **it is meaningless to ask whether the Maxwell action $S[A_\mu]$ is invariant under the above duality transformations of the field strengths since there is no variation of A_μ that yields these variations.**

Explicit form of the duality transformation of the vector potential under em duality

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As we just argued, this is the best one can hope for.

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One choose the duality transformations of the vector potential such that these non-local terms are non-local only in space, where the inverse Laplacian $\triangle^{-1}$ can be given a meaning. Terms non local in time (and $\square^{-1}$) are much more tricky. One can also use the gauge ambiguity in the definition of δA_μ in such a way that $\delta A_0 = 0$.

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The duality transformation of the vector potential is then given (up to a residual gauge symmetry) by

$$\delta A_0 = 0, \quad \delta A^i = -\epsilon \triangle^{-1} \left(\epsilon^{ijk} \partial_j F_{0k} \right).$$

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Similarly, $\delta E_i = -\delta \dot{A}_i = \epsilon B_i - \epsilon \triangle^{-1} (\epsilon_{ijk} \partial^j \partial_\mu F^{\mu k})$.

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It is therefore a genuine Noether symmetry (with Noether charge etc).

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If, besides the standard “magnetic" vector potential defined through,

$$\vec{B} \equiv \vec{B}_1 = \vec{\nabla} \times \vec{A}_1,$$

one introduces an additional vector potential $\vec{A}_2$ through,

$$\vec{E} \equiv \vec{B}_2 = \vec{\nabla} \times \vec{A}_2,$$

one may rewrite the standard Maxwell action in terms of the two potentials A^a as

$$S = \frac{1}{2} \int dx^0 d^3x \left(\epsilon_{ab} \vec{B}^a \cdot \dot{\vec{A}}^b - \delta_{ab} \vec{B}^a \cdot \vec{B}^b \right).$$

Here, ϵ_{ab} is given by $\epsilon_{ab} = -\epsilon_{ba}$, $\epsilon_{12} = +1$.

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To conclude : the “proof" using the standard form of the em duality transformations that the second order Maxwell action $S[A_\mu] = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}$ is not invariant under duality transformations – and thus that duality is only an “on-shell symmetry" – is incorrect because it is based on a form of the duality transformations that is inconsistent with the existence of the dynamical variable A_μ. A consistent set of transformations leaves the action invariant.

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The duality transformations of the field strengths are given by the standard ones plus correction terms that vanish on-shell but are necessary in order to have $\delta F^I = d\delta A^I$. These correction terms are non-covariant and non-local in space. They are present whenever the transformations mix electric and magnetic fields.

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In this formulation, Poincaré invariance is not manifest, however. More on this later.

Duality invariance of the Einstein equations

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The Riemann tensor

$$R_{\lambda\mu\rho\sigma} = -\frac{1}{2}\left(\partial_\lambda\partial_\rho h_{\mu\sigma} - \partial_\mu\partial_\rho h_{\lambda\sigma} - \partial_\lambda\partial_\sigma h_{\mu\rho} + \partial_\mu\partial_\sigma h_{\lambda\rho}\right)$$

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This transformation rotates the Schwarschild mass into the Taub-NUT parameter N.

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Is this also a symmetry of the Pauli-Fierz action ?

$$S[h_{\mu\nu}] = -\frac{1}{4}\int d^4x \left[\partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu} - 2\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_{\ \nu} + 2\partial^\mu h \partial^\nu h_{\mu\nu} - \partial^\mu h \partial_\mu h\right].$$

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It is obtained by starting from the first-order (Hamiltonian) action and solving the constraints.

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For instance, the momentum constraint $\partial_i \pi^{ij} = 0$ is solved by

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Both are invariant under

$$\delta Z^a_{ij} = \partial_i \xi^a_j + \partial_j \xi^a_i + 2\epsilon^a \delta_{ij}$$

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$$H = \int d^3x \left(4 R^a_{ij} R^{bij} - \frac{3}{2} R^a R^b \right) \delta_{ab}.$$

Here, R^a_{ij} is the Ricci tensor constructed out of the prepotential Z^a_{ij}.

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Henneaux-Teitelboim 2005.

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We consider here explicitly the gravitational case.

Twisted self-duality for gravity

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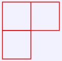
For definiteness, consider $D=5$. In that case, the “dual graviton" is described by a tensor $T_{\alpha\beta\gamma}$ of mixed symmetry type described by the Young tableau



$T_{\alpha\beta\gamma} = T_{[\alpha\beta]\gamma}$, $T_{[\alpha\beta\gamma]} = 0$.

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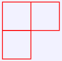

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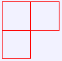

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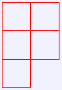

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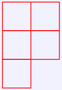

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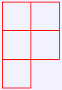

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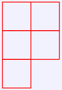

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$$E_{\alpha_1\alpha_2\beta} = 0$$

for the “Ricci tensor" $E_{\alpha_1\alpha_2\beta}$.

Twisted self-duality for gravity

The Einstein equations $R_{\mu\nu} = 0$ for the Riemann tensor $R_{\mu\nu\alpha\beta}[h]$ imply that the dual Riemann tensor $E_{\beta_1\beta_2\beta_3\rho_1\rho_2}$, defined by

$$\begin{aligned} E_{\beta_1\beta_2\beta_3\rho_1\rho_2} &= \frac{1}{2!}\epsilon_{\beta_1\beta_2\beta_3\alpha_1\alpha_2} R^{\alpha_1\alpha_2}{}_{\rho_1\rho_2} \\ R_{\alpha_1\alpha_2\rho_1\rho_2} &= -\frac{1}{3!}\epsilon_{\alpha_1\alpha_2\beta_1\beta_2\beta_3} E^{\beta_1\beta_2\beta_3}{}_{\rho_1\rho_2} \end{aligned}$$

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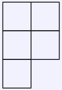

Here, $h_{\alpha\beta}$ is the spin-2 (Pauli-Fierz) field.

Twisted self-duality conditions for gravity

Furthermore, (i) the tensor $E_{\beta_1\beta_2\beta_3\rho_1\rho_2}$ obeys the differential identities $\partial_{[\beta_0} E_{\beta_1\beta_2\beta_3]\rho_1\rho_2} = 0$, $E_{\beta_1\beta_2\beta_3[\rho_1\rho_2,\rho_3]} = 0$ that guarantee the existence of a tensor $T_{\alpha\beta\mu}$ such that

$$E_{\beta_1\beta_2\beta_3\rho_1\rho_2} = E_{\beta_1\beta_2\beta_3\rho_1\rho_2}[T];$$

and (ii) the field equations for the dual tensor $T_{\alpha\beta\mu}$ are satisfied.

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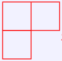
- Let $h_{\mu\nu}$ and $T_{\alpha\beta\mu}$ be tensor fields of respective Young symmetry types  and , and let $R_{\alpha_1\alpha_2\rho_1\rho_2}[h]$ and $E_{\beta_1\beta_2\beta_3\rho_1\rho_2}[T]$ be the corresponding gauge-invariant curvatures.

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The “twisted self-duality conditions", which express that E is the dual of R (we drop indices)

$$R = -\,{}^*E, \quad E = {}^*R,$$

or, in matrix notations,

$$\mathfrak{R} = \mathscr{S}\,{}^*\mathfrak{R},$$

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imply that $h_{\mu\nu}$ and $T_{\alpha\beta\mu}$ are both solutions of the linearized Einstein equations and the Curtright equations,

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- One can define electric and magnetic fields for $h_{\alpha\beta}$ and $T_{\alpha\beta\gamma}$. The twisted self-duality conditions are equivalent to $\mathscr{B}_{ijrs}[T] = \mathscr{E}_{ijrs}[h]$, $\mathscr{B}_{ijr}[h] = -\mathscr{E}_{ijr}[T]$.

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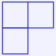
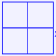
  This step introduces “prepotentials", of respective Young symmetry type  and , which are again canonically conjugate.

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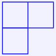
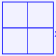
  This step introduces “prepotentials", of respective Young symmetry type  and , which are again canonically conjugate.

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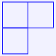
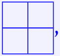
  This step introduces “prepotentials", of respective Young symmetry type  and , which are again canonically conjugate.

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- The details can be found in Bunster-Henneaux-Hörtner 2013.

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E_{10} and gravitational duality

Marc Henneaux

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THANK YOU !