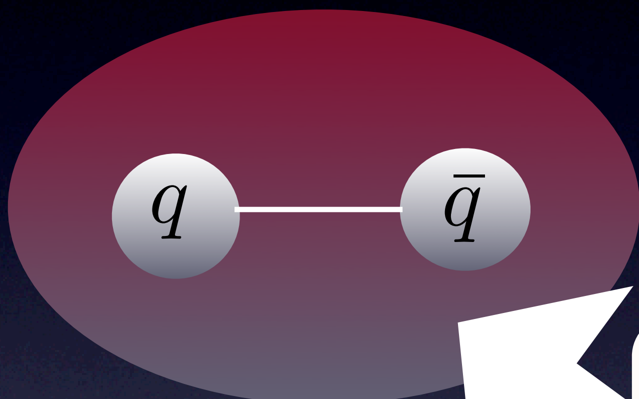
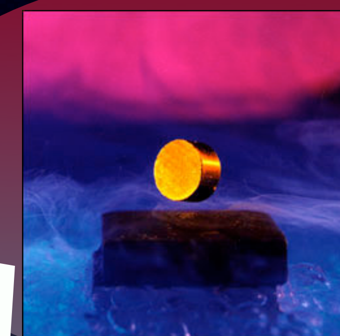
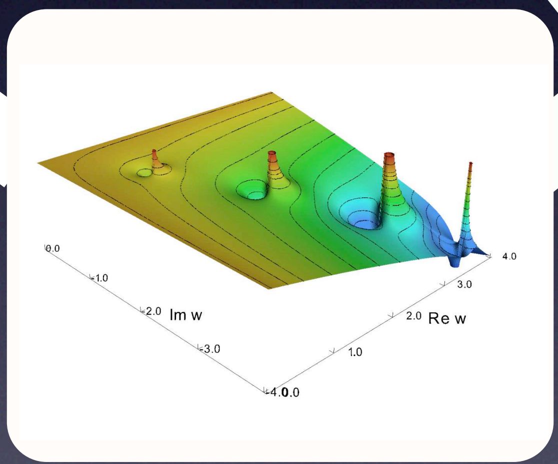
Hydrodynamics, Quasinormal Modes & Holographic SSB

‘Fluid/Gravity Correspondence’ Workshop, ASC Munich, September 7th 2009







by Matthias Kaminski

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in Collaboration with Irene Amado, Martin Ammon, Johanna Erdmenger, Constantin Greubel, Patrick Kerner, Karl Landsteiner, Francisco Pena

Navigator

I. Invitation: Hydrodynamics & QNMs

II. Holographic Spont. Symm. Breaking (SSB)

(a) Hydrodynamic predictions at SSB

(b) Determinant Method for coupled QNMs

(c) QNMs, sound & critical exponents

III. Holographic Quark-Gluon-Plasma

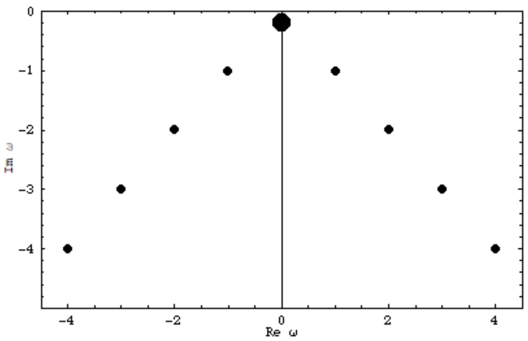
(a) QNMs of D3/D7

(b) Open questions

IV. Conclusion

I. Invitation: : *The Importance of Being Quasinormal*

Derived from black hole fluctuations (black hole ringing) quasinormal modes (QNM) encode dynamical information.



Defined as modes with Dirichlet b.c. at the AdS boundary, infalling at horizon.
(to be refined)
Lowest QNMs correspond to hydrodynamic modes (spectral function, residue).
[Berti,Cardoso,Starinets 0905.2975]

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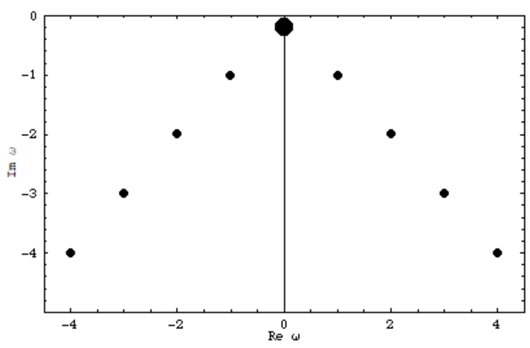

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Study of quasinormal modes (QNMs) has (at least) two virtues:

1. Phenomenology:
   - hydrodynamics of strongly coupled systems (QGP, non-BCS SC)
   - transport coefficients *[talk by A. Starinets]*
   - scaling exponents/dynamic critical phenomena
   - Fermi surface *[talk by S. Hartnoll]*

[talks by D. Mateos and C.P. Herzog]

2. Universal Results:
   - real world black hole “eigenfrequencies” (not AdS)
   - universal transport coeffs, scaling exponents / universality classes
   - specific models can trigger universal results, e.g. anomalous coeffs

[Erdmenger,Haack, M.K. ,Yarom 0809.2488] *[cf. talk by Loganayagam]*
[Son,Surowka 0906.5044]

II. Holo SSB: (a) Hydrodynamic predictions

Hydrodynamic modes from conserved charges or phases of order parameters (Goldstone bosons). Phenomenological, thus 'universal'.

Conservation and modified Fick's law equation

$$\dot{n} + \vec{\nabla} \cdot \vec{j} = 0 \qquad \vec{j} = -D\vec{\nabla}n + \vec{\sigma} \cdot E - \tau\dot{\vec{j}}$$

$n,\ \vec{j},\ \tau$ are the charge density, spatial current and relaxation time.

Vanishing external field leaves the "causal" diffusion equation

$$0 = \partial_t n - D\nabla^2 n + \tau\partial_t{}^2 n$$

At spontaneous symmetry breaking currents take infinite time to relax:

$$\tau \to \infty \qquad D \to \tau\hat{D} \qquad \sigma \longrightarrow \tau\hat{\sigma} \qquad (\sigma \sim D)$$

II. Holo SSB: (a) Hydrodynamic predictions 2

Space-time dependence of current

$$\vec{j} = \vec{j}_0 \, e^{-\frac{t}{\tau}} e^{-i\omega t + ikx}$$

Time-derivative of modified Fick's law

$$\partial_t j_k = D\nabla^2 j_k + \sigma_{lk}\partial_t E_l - \tau {\partial_t}^2 j_k$$

Longitudinal current only

$$\partial_t j_L = D\nabla^2 j_L + \sigma\partial_t E_L - \tau {\partial_t}^2 j_L$$

$$E_L = \partial_t A_L$$

II. Holo SSB: (a) Hydrodynamic predictions 2

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In Fourier space

$$j_L = \frac{-\sigma\omega^2 A_L}{(-i\omega - \frac{1}{\tau}) + Dk^2 - \tau\omega^2 + 2i\omega + \frac{1}{\tau}} = \frac{-\tau\hat{\sigma}\omega^2 A_L}{(-i\omega - \frac{1}{\tau}) + \tau\hat{D}k^2 - \tau\omega^2 + 2i\omega + \frac{1}{\tau}}$$

➔ Longit. current two point correlator with sound pole if $\tau \longrightarrow \infty$

$$\boxed{\langle j_L \, j_L \rangle = \frac{\hat{\sigma}\omega^2}{\omega^2 - \hat{D}k^2} \qquad \hat{D} = {v_s}^2}$$

Diffusion pole converted to sound pole?

➔ Infinite DC conductivity

Peak? Gap?

$$\kappa(\omega) = \frac{1}{\omega}\langle j_L \, j_L \rangle_{k=0} \sim \frac{n_s}{\omega + i\epsilon} \sim n_s P(\frac{1}{\omega}) + i n_s \pi \delta(\omega)$$

II. Holo SSB: (b) Determinant Method

Example: Condensing scalar model in planar Schwarzschild AdS b.h. in probe limit

[Hartnoll, Herzog, Horowitz'08]

[see talk by C.P.Herzog]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}(dx^2 + dy^2) \qquad f(r) = \frac{r^2}{L^2} - \frac{M}{r}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m^2\Psi\bar{\Psi} - (\partial_\mu\Psi - iA_\mu\Psi)(\partial^\mu\bar{\Psi} + iA^\mu\bar{\Psi})$$

with tachyonic mass $m^2 = -2/L^2$ and gauge such that scalar field real.

Scalar field and temporal component of gauge field EOMs (background)

$$\Psi'' + (\frac{f'}{f} + \frac{2}{\rho})\Psi' + \frac{\Phi^2}{f^2}\Psi + \frac{2}{L^2 f}\Psi = 0\,,$$

$$\Phi'' + \frac{2}{\rho}\Phi' - \frac{2\Psi^2}{f}\Phi = 0\,.$$

AdS boundary asymptotics

$$\Phi = \bar{\mu} - \frac{\bar{n}}{\rho} + O(\frac{1}{\rho^2})\,,$$

$$\Psi = \frac{\psi_1}{\rho} + \frac{\psi_2}{\rho^2} + O(\frac{1}{\rho^2})\,.$$

theory 1: ψ_1 source for dim 2 op.

theory 2: ψ_2 source for dim 1 op.

II. Holo SSB: (b) Determinant Method 2

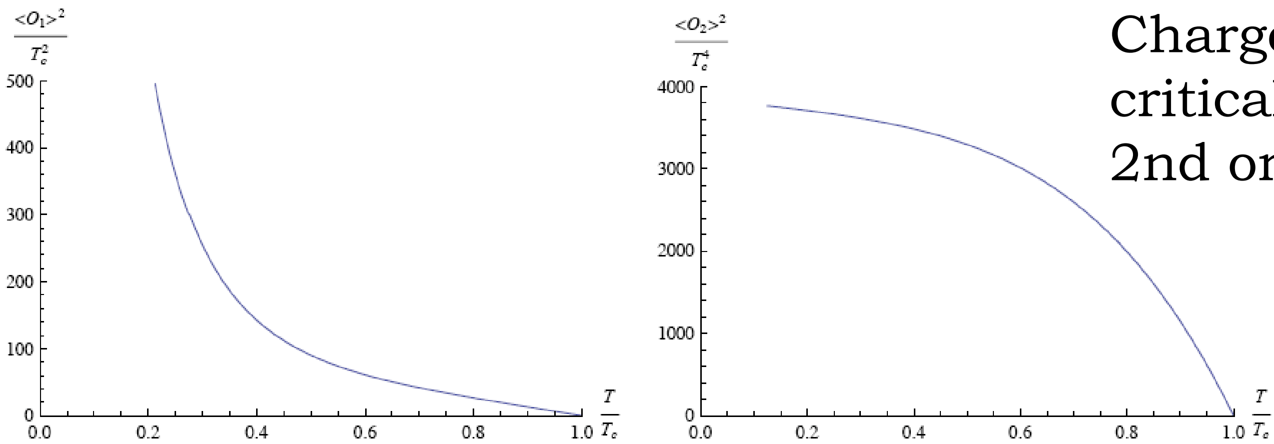


Figure 1: The condensates as function of the temperature in the two possible theories.

Charged scalar condenses below critical temperature in both theories. 2nd order superfluid transition.

Add fluctuations

$$\Psi = \psi(\rho) + \sigma(\rho, t, x) + i\eta(\rho, t, x)\,,$$

$$A_\mu = \mathcal{A}_m(\rho) + a_\mu(\rho, t, x)\,.$$

where Goldstone boson appears as phase of order parameter, i.e. η

EOMs in unbroken phase (dynamics):

$$0 = \Psi'' + (\frac{f'}{f} + \frac{2}{\rho})\Psi' + (\frac{(\Phi+\omega)^2}{f^2} + \frac{2}{f} - \frac{k^2}{f\rho^2})\Psi\,, \quad \omega = \frac{3\omega_{ph}}{4\pi T} \quad , \quad k = \frac{3k_{ph}}{4\pi T}\,.$$

$$0 = a_t'' + \frac{2}{\rho}a_t' - \frac{k^2}{\rho^2}a_t - \frac{\omega k}{f\rho^2}a_x\,,$$

$$0 = a_x'' + \frac{f'}{f}a_x' + \frac{\omega^2}{f^2}a_x + \frac{\omega k}{f\rho^2}a_t\,,$$

with constraint $0 = \frac{\omega}{f}a_t' + \frac{k}{\rho^2}a_x'\,.$

II. Holo SSB: (b) Determinant Method 3

Coupled EOMs in broken phase (dynamics): *[Amado, MK, Landsteiner 0903.2209]*

$$0 = f\eta'' + \left(f' + \frac{2f}{\rho}\right)\eta' + \left(\frac{\phi^2}{f} + \frac{2}{L^2} + \frac{\omega^2}{f} - \frac{k^2}{\rho^2}\right)\eta - \frac{2i\omega\phi}{f}\sigma - \frac{i\omega\psi}{f}a_t - \frac{ik\psi}{r^2}a_x\,,$$

$$0 = f\sigma'' + \left(f' + \frac{2f}{\rho}\right)\sigma' + \left(\frac{\phi^2}{f} + \frac{2}{L^2} + \frac{\omega^2}{f} - \frac{k^2}{\rho^2}\right)\sigma + \frac{2\phi\psi}{f}a_t + \frac{2i\omega\phi}{f}\eta\,,$$

$$0 = f{a_t}'' + \frac{2f}{\rho}{a_t}' - \left(\frac{k^2}{\rho^2} + 2\psi^2\right)a_t - \frac{\omega k}{\rho^2}a_x - 2i\omega\psi\,\eta - 4\psi\phi\,\sigma\,,$$

$$0 = f{a_x}'' + f'{a_x}' + \left(\frac{\omega^2}{f} - 2\psi^2\right)a_x + \frac{\omega k}{f}a_t + 2ik\psi\,\eta\,.$$

Transverse to momentum, thus decoupled

$$0 = f\,{a_y}'' + f'{a_y}' + \left(\frac{\omega^2}{f} - \frac{k^2}{\rho^2} - 2\psi^2\right)a_y$$

Constraint $\quad \frac{\omega}{f}{a_t}' + \frac{k}{\rho^2}{a_x}' = 2i\left(\psi'\,\eta - \psi\,\eta'\right)\,,$

Second example:
QNMs of D3/D7-system at finite momentum and baryon density
[MK, Landsteiner,Mas, Shock,Tarrio 0910.xxxx]

II. Holo SSB: (b) Determinant Method 4

Quasinormal modes of coupled systems

$$a_t = \mathfrak{b}_1 a_t{}^I + \mathfrak{b}_2 a_t{}^{II}$$

$$a_x = \mathfrak{b}_1 a_x{}^I + \mathfrak{b}_2 a_x{}^{II}$$

where I and II are distinct sets of boundary conditions.

QNM condition at AdS boundary

$$\lim_{r\to\infty} a_t = 0 = \lim_{r\to\infty} a_x$$

Normalize and plug in for coeff

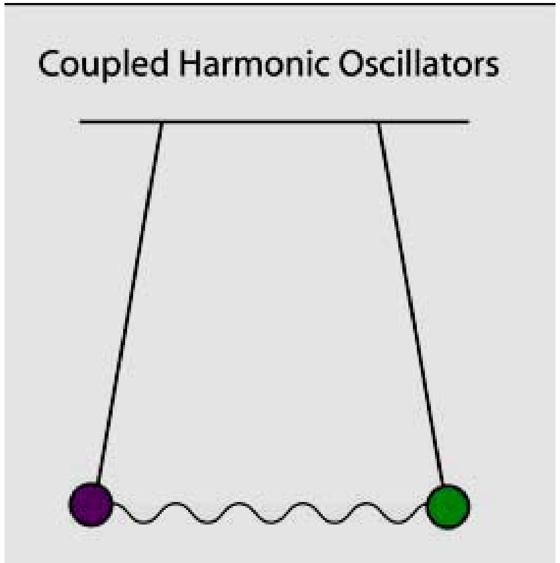
$$0 = -\frac{a_t{}^{II}}{a_t{}^{I}} a_x{}^I + a_x{}^{II} \Big|_{r\to\infty}$$

$$0 = det \begin{pmatrix} a_t{}^I & a_t{}^{II} \\ a_x{}^I & a_x{}^{II} \end{pmatrix}_{r\to\infty}$$

Example: $a_t{}^{II} = -\omega\alpha\,, \quad a_x{}^{II} = k\alpha$

$$0 = \omega\alpha\, a_x{}^I + k\alpha\, a_t{}^I \equiv \alpha E_x$$

Analog



Coupling an oscillator with eigenfrequency ω_1 to another with frequency ω_2, there are two new eigenmodes of the coupled system, not assigned to only one of the oscillators.

II. Holo SSB: (b) Determinant Method 5

General formalism switching on specific operators

$$\begin{pmatrix} c_1^1 & \dots & c_n^1 \\ & \dots & \\ c_1^n & \dots & c_n^n \end{pmatrix} \cdot \underbrace{\begin{pmatrix} {\phi_1}^I & \dots & {\phi_1}^{I_n} \\ & \dots & \\ {\phi_n}^I & \dots & {\phi_n}^{I_n} \end{pmatrix}_{r\to\infty}}_{\mathfrak{F}} = \text{diagonal}(1,\dots,1)$$

➔ retarded Green function

Set right hand side to zero:

General QNM condition

$$det\, \mathfrak{F}_{r\to\infty} = 0$$

QNM corresponding to poles in holographic Green functions are zeroes of the determinant of field values at the AdS boundary for a maximal set of linearly independent solutions (infalling at horizon).

Advantages

- no explicit trafo to gauge-invariant fields needed
- elegant formulation including all fields simultaneously
- works for any coupled system of gravitational fluctuations
- need not compute full Green function (but possible)
- coupled EOMs along r dual to operator mixing along RG-flow, cutoff!

II. Holo SSB: (c) QNMs, sound, crit. exp.

Unbroken phase

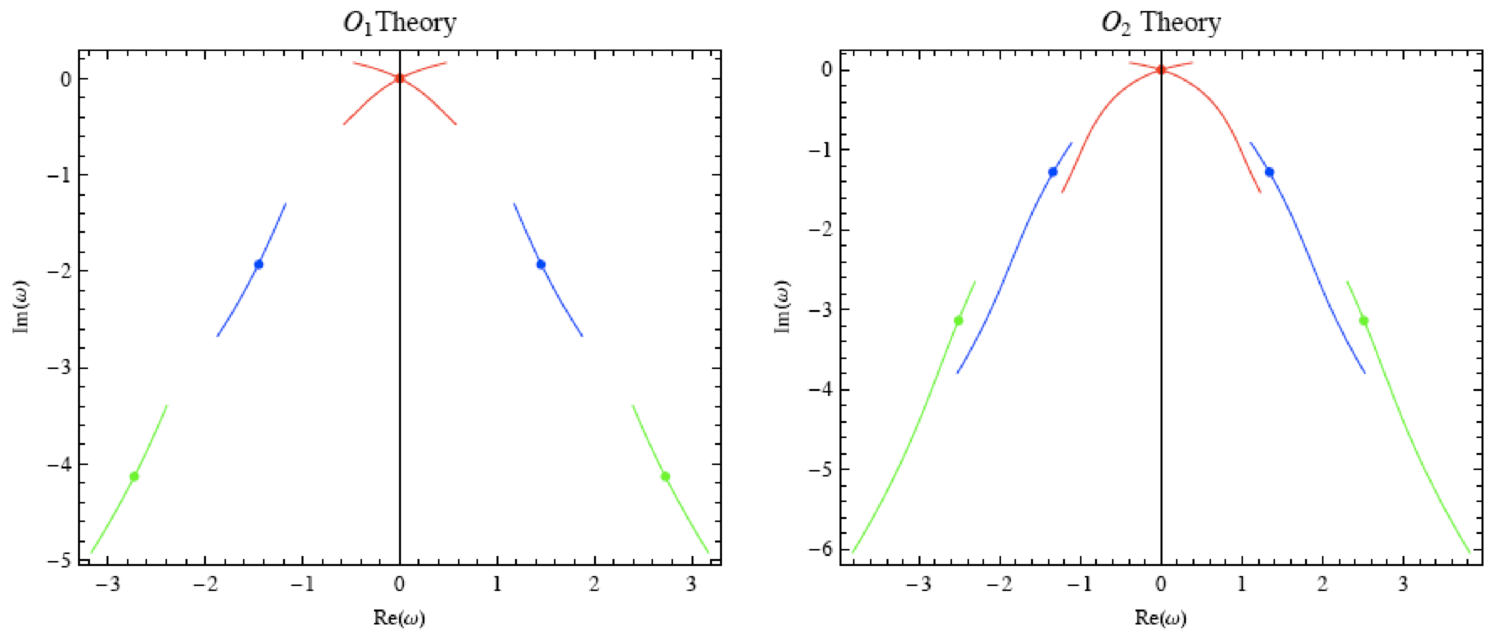


Figure 2: Lowest scalar quasinormal frequencies as a function of the temperature and at momentum $k = 0$, from $T/T_c = \infty$ to $T/T_c = 0.81$ in the O_2 theory (right) and to $T/T_c = 0.56$ in the O_1 theory (left). The dots correspond to the critical point $T/T_c = 1$ where the phase transition takes place. Red, blue and green correspond to first, second and third mode respectively.

Identify the Goldstone boson (hydro modes' residue would vanish)

$$\chi = \lim_{k,\omega \to 0} \langle \rho\rho \rangle = \lim_{k,\omega \to 0} \frac{i\sigma k^2}{\omega + iDk^2} = \frac{\sigma}{D}\,, \qquad \chi_{\bar{O}_i O_i} = \lim_{k,\omega \to 0} \langle \bar{O}_i O_i \rangle = \lim_{k,\omega \to 0} \frac{R_i(k, T_c)}{\omega - \omega_H(k, T_c)} \to \infty$$

Gauge field dynamics is temperature independent. Diffusion mode.

II. Holo SSB: (c) QNMs, sound, crit. exp.

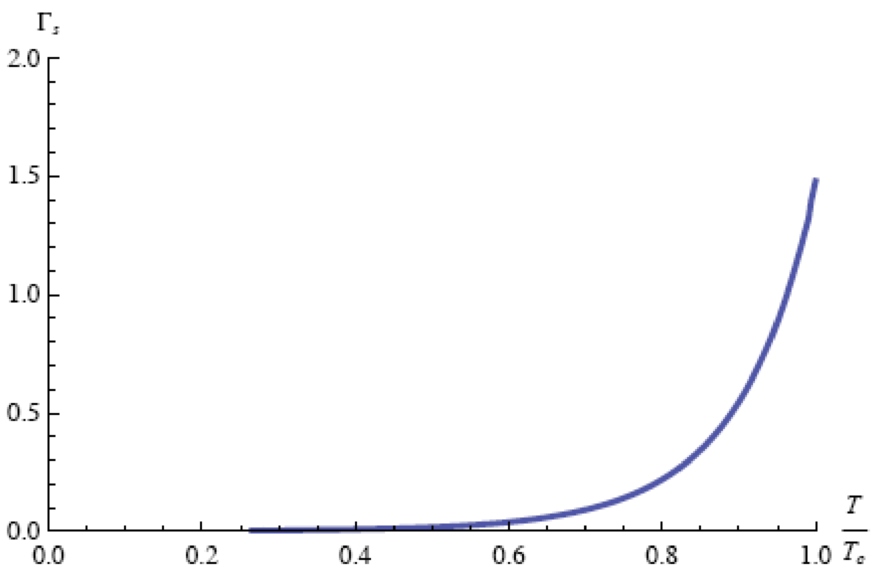
Broken phase results

- tachyons turn into sound modes (Goldstone modes)
- diffusion mode does NOT split into sound modes
- speed of 'fourth' sound agrees with *[Herzog,Kovtun,Son 0809.4870]*
- speed asymptotes to 1/3 at zero T (or 1/2 for dimension 1)*[Yarom 0903.1353]*
- similarly in p-wave setup *[Ammon,Erdmenger, MK, Kerner 0810.2316]*

[see talk by J.K.Erdmenger]

Sound speed and attenuation

$$\omega = v_s k - i\Gamma_s k^2$$



$$v_s^2 \approx 1.9\left(1 - \frac{T}{T_c}\right) \qquad \mathrm{O}_1 - \text{Theory}\,,$$

$$v_s^2 \approx 2.8\left(1 - \frac{T}{T_c}\right) \qquad \mathrm{O}_2 - \text{Theory}\,.$$

Attenuation finite at transition

$$\Gamma_s = 1.87T_c \quad \text{at} \quad T = 0.9991T_c \qquad \mathrm{O}_1 - \text{Theory}\,,$$

$$\Gamma_s = 1.48T_c \quad \text{at} \quad T = 0.9998T_c \qquad \mathrm{O}_2 - \text{Theory}\,.$$

[cf. Buchel,Pagnutti '09]

II. Holo SSB: (c) QNMs, sound, crit. exp.

Pseudo-diffusion mode

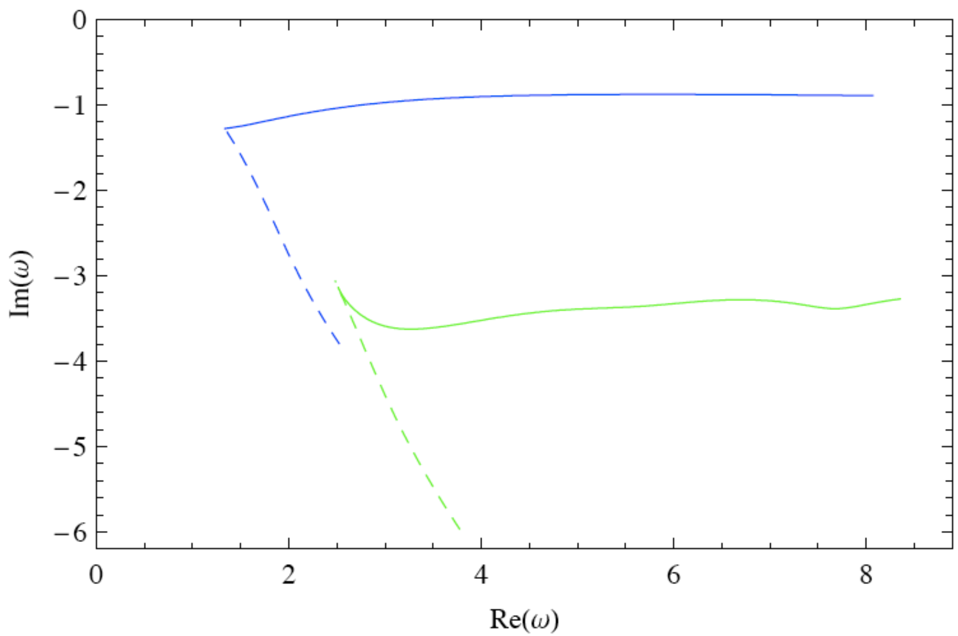
$$\omega = -iDk^2 - i\gamma(T)\,,$$

$$\gamma \approx 15.4 T_c \left(1 - \frac{T}{T_c}\right) \qquad \mathrm{O}_1 - \text{Theory}\,,$$

$$\gamma \approx 8.1 T_c \left(1 - \frac{T}{T_c}\right) \qquad \mathrm{O}_2 - \text{Theory}\,.$$

Diffusion mode becomes gapped in broken phase.
Gap scales linearly with condensate.

Higher QNMs



Unbroken phase scalars: dashed
Broken phase: solid

Parallel motion related to conductivity gap?
Different from usual low-T.

Navigator

✓ Invitation: Hydrodynamics & QNMs

✓ Holographic Spont. Symm. Breaking (SSB)

(a) Hydrodynamic predictions at SSB

(b) Determinant Method for coupled QNMs

(c) QNMs, sound & critical exponents

II. Holographic Quark-Gluon-Plasma

(a) QNMs of D3/D7

(b) Open questions

III. Conclusion

III. Holographic Quark Gluon Plasma

Second Example: Coupled fluctuation EOMs at finite density and momentum

[MK,Landsteiner,Mas,Shock,Tarrio 0910.xxxx]

- D_3-branes

[Karch, Katz; hep-th/0205236]

III. Holographic Quark Gluon Plasma

Second Example: Coupled fluctuation EOMs at finite density and momentum

[MK,Landsteiner,Mas,Shock,Tarrio 0910.xxxx]

D_7-branes

'Minkowski' embedding

• D_3-branes

[Karch, Katz; hep-th/0205236]

III. Holographic Quark Gluon Plasma

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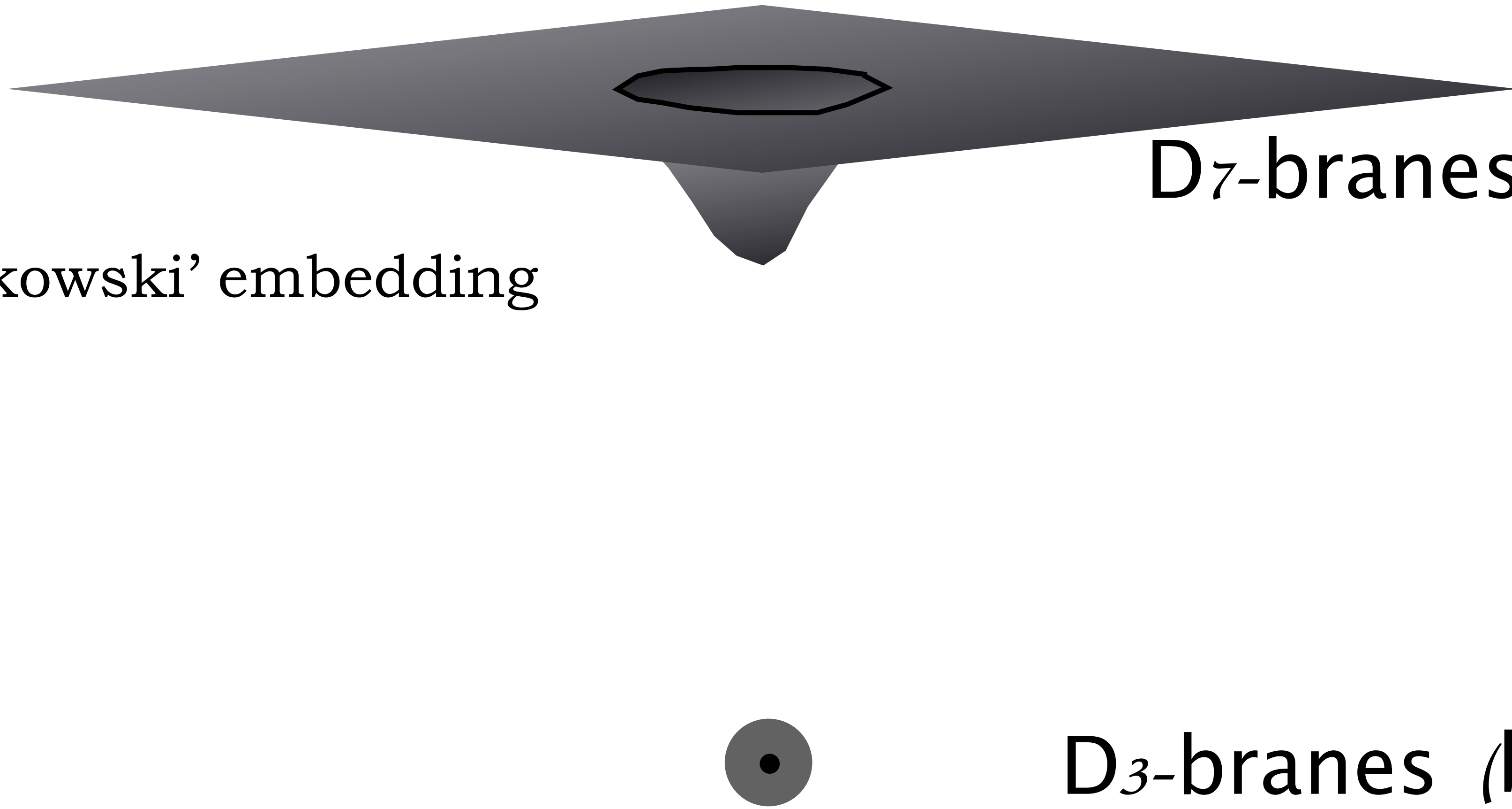

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III. Holographic Quark Gluon Plasma

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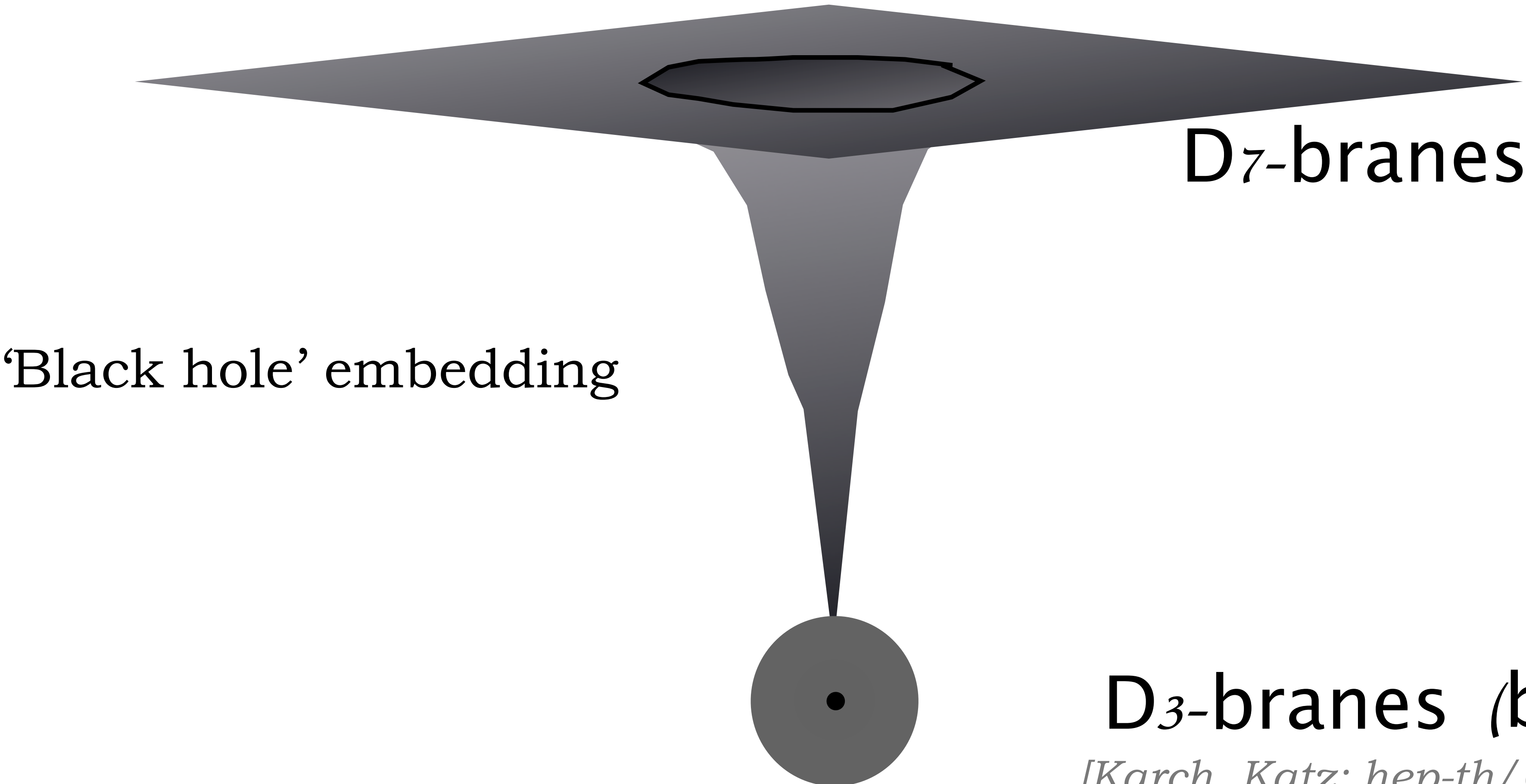

[Karch, Katz; hep-th/0205236]

III. Holographic Quark Gluon Plasma

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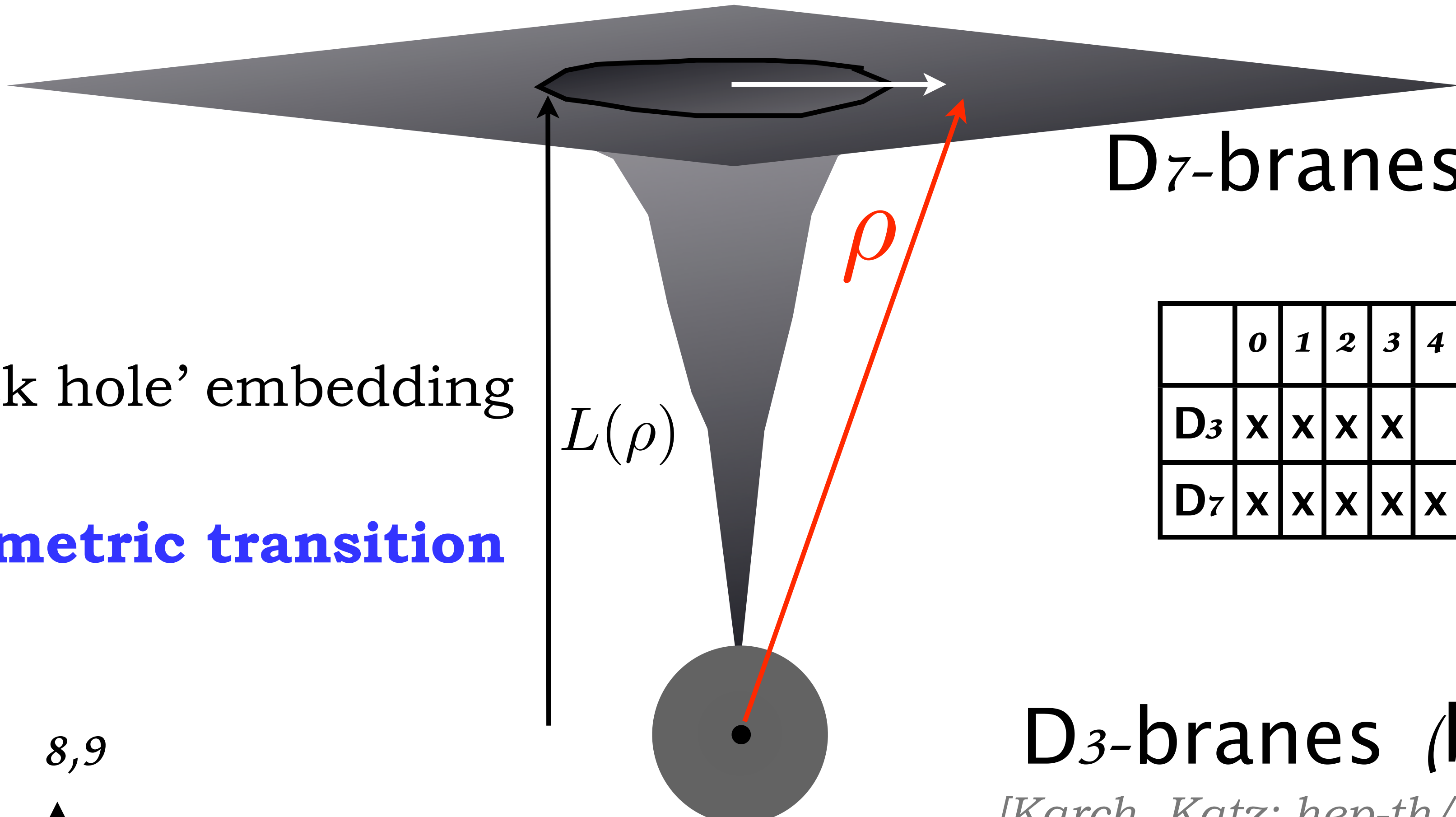
'Black hole' embedding

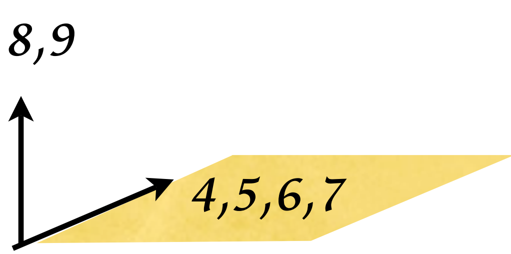
Geometric transition



	0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
D3	x	x	x	x						
D7	x	x	x	x	x	x	x	x		

D3-branes (black)

[Karch, Katz; hep-th/0205236]



III. Holographic Quark Gluon Plasma

Second Example: Coupled fluctuation EOMs at finite density and momentum

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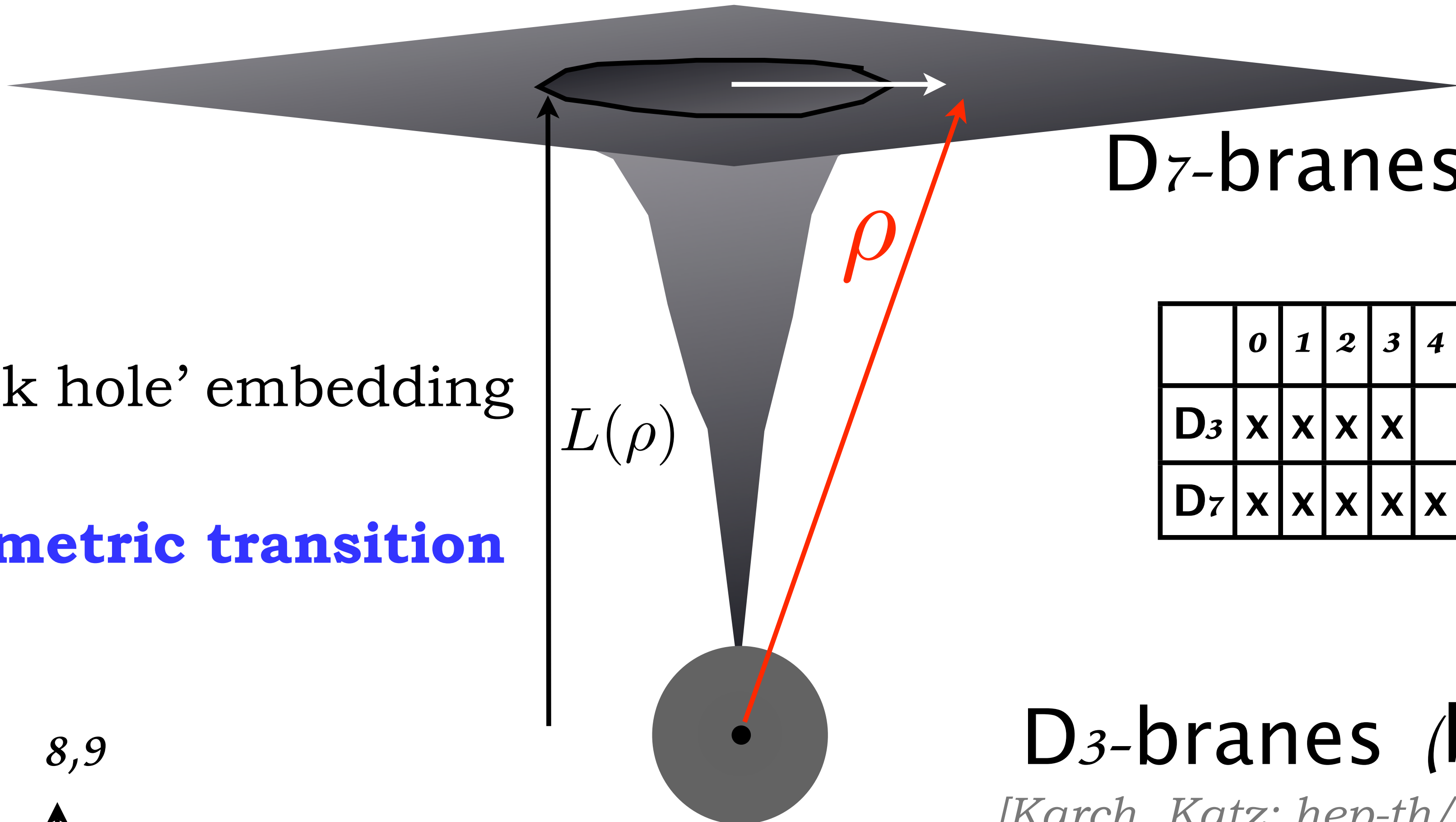

'Black hole' embedding

Geometric transition

	0	1	2	3	4	5	6	7	8	9
D3	x	x	x	x						
D7	x	x	x	x	x	x	x	x		

D3-branes (black)

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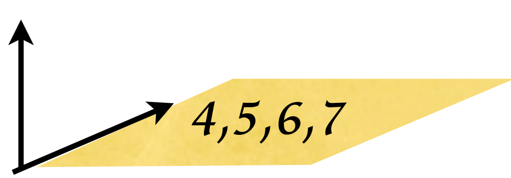


Chemical potential: $\hat{A}_\mu = \delta_{\mu 0} A_0 + \tilde{A}_\mu$

[Nakamura et al., hep-th/0611021]

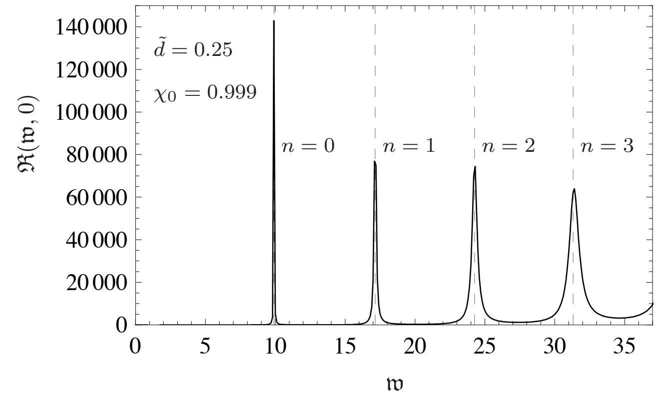
[Myers et al., hep-th/0611099]

III. Holographic Quark Gluon Plasma

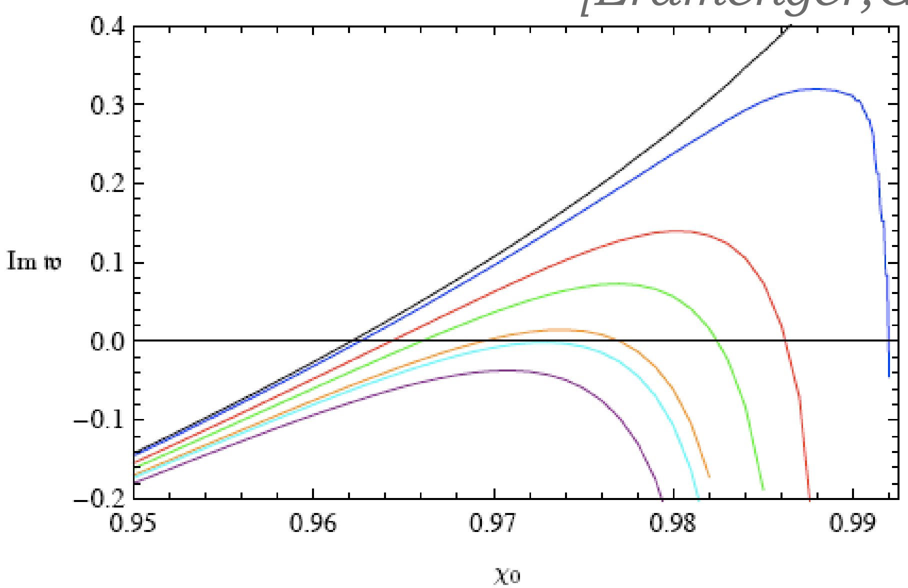
Scalar spectrum at zero density contains tachyon.
Spectral functions in 'undercooled phase' show no clear meson resonances.

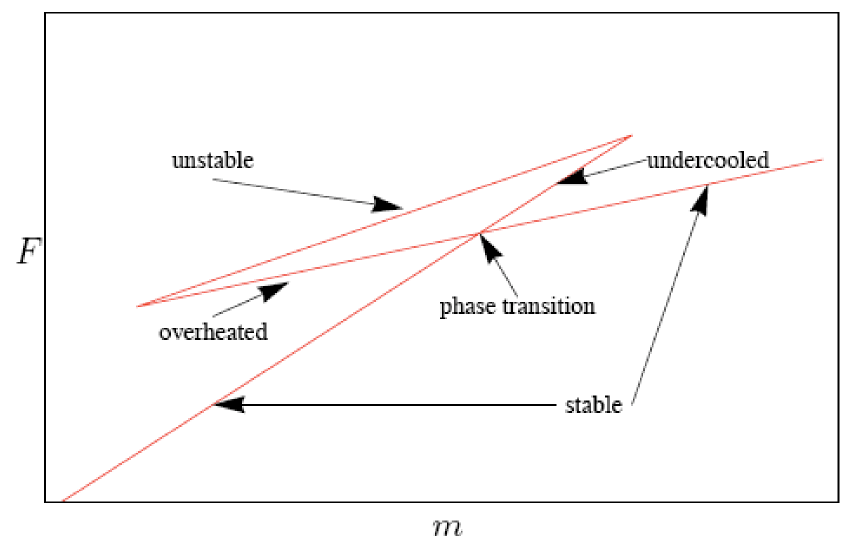
[see talk by D.Mateos]

Meson spectra at finite density show meson resonances (associated to QNMs).
[Erdmenger, MK, Rust 0710.0334]



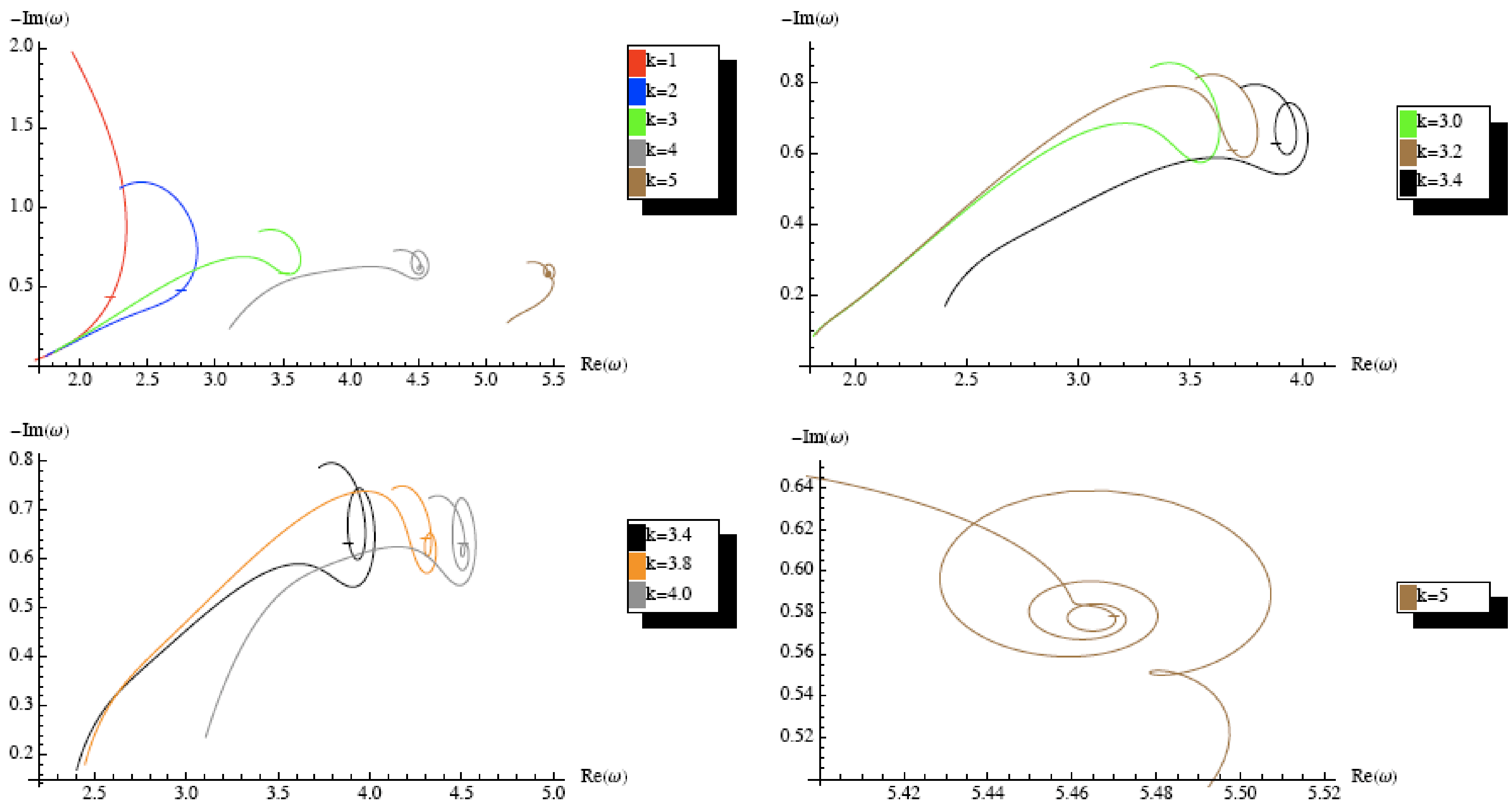
Cure the known instability with finite baryon density
[Erdmenger,Greubel,MK,Kerner,Landsteiner,Pena 0910.xxxx]





III. Holographic Quark Gluon Plasma

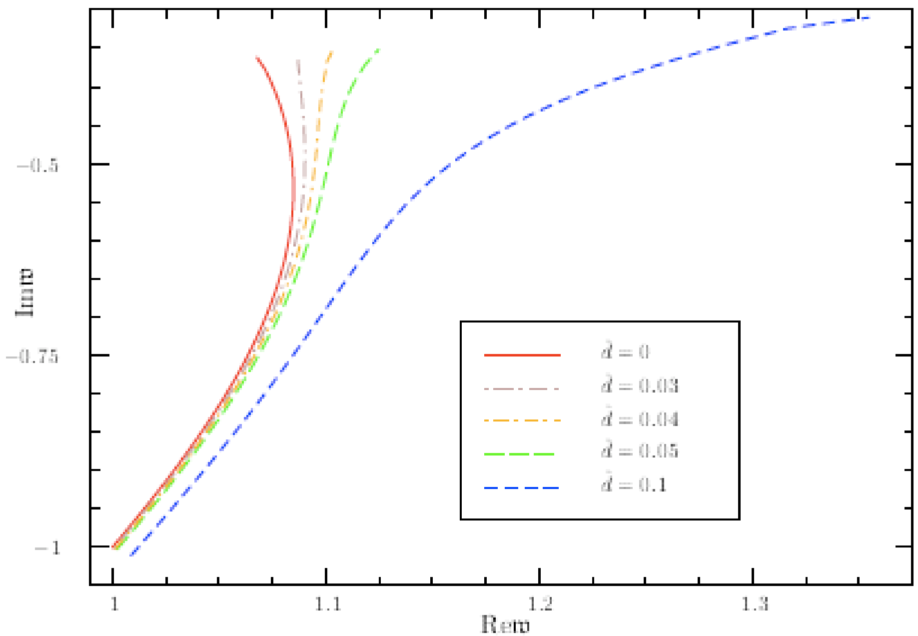
First longitudinal (transverse similar) vector QNM at increasing momenta k, vanishing d *[see talk by D.Mateos]*



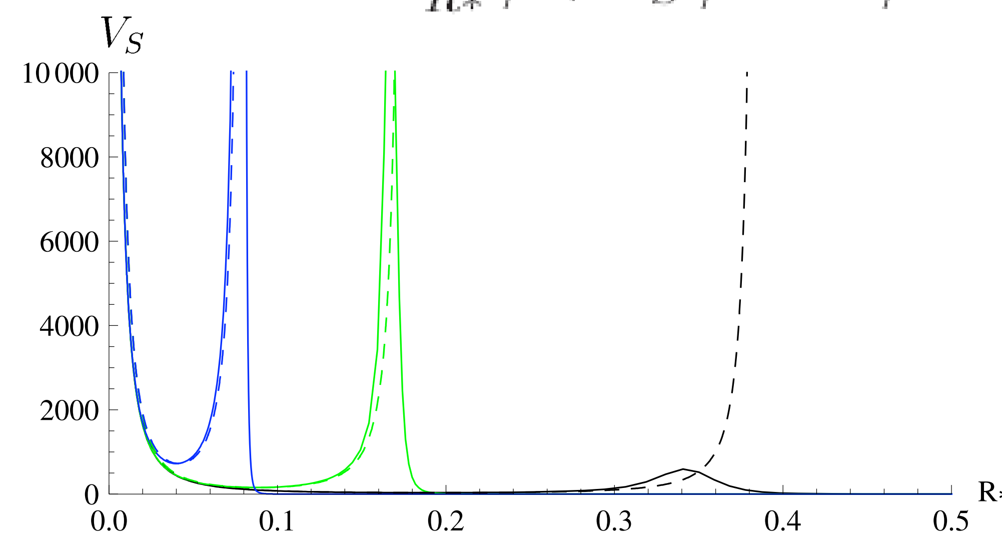
- Meson masses pile up at ‘attractor’ frequencies, jumping with k
- Number of spirals correlated to (unphysical) ‘attractor’ frequencies?!

III. Holographic Quark Gluon Plasma

First transverse vector QNM and Schroedinger potential
at finite density



$$-\partial^2_{R*}\psi + V_S\psi = E\psi\,.$$



AdS boundary

horizon

Right-motion:
Potential barrier forms 'near horizon' (where corresponding Minkowski ends).
Less leakage into black hole.

Left-motion:
Effective description with damped harmonic oscillator?

III. Holographic Quark Gluon Plasma

Hydrodynamic to collisionless crossover

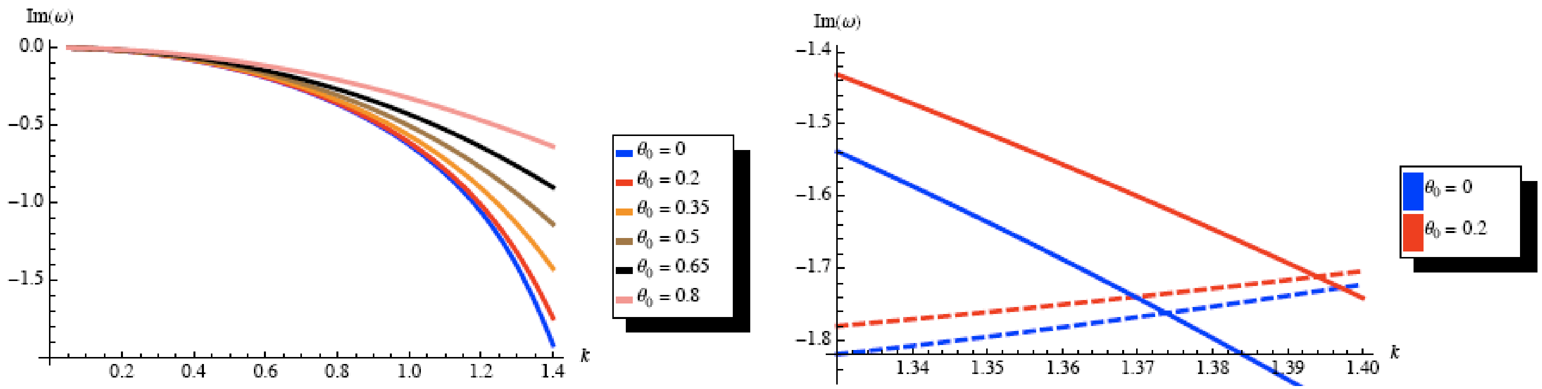
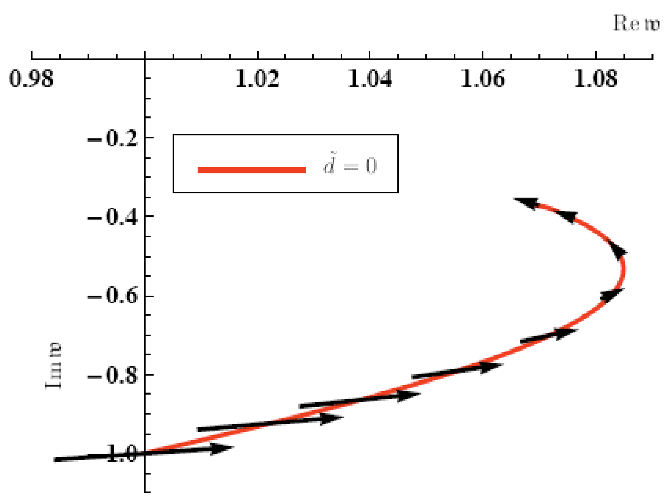


Figure 17: Right: The dispersion relation for the diffusion mode at distinct values for the mass/temperature parameter θ_0. Left: Intersection of the diffusive mode with the imaginary part of the first longitudinal qusinormal mode. The intersection point moves to larger values of k, but to smaller values of Im(ω) as the mass/temperature parameter θ_0 is increased.

Residues



III. Holographic Quark Gluon Plasma

Hydrodynamic to collisionless crossover

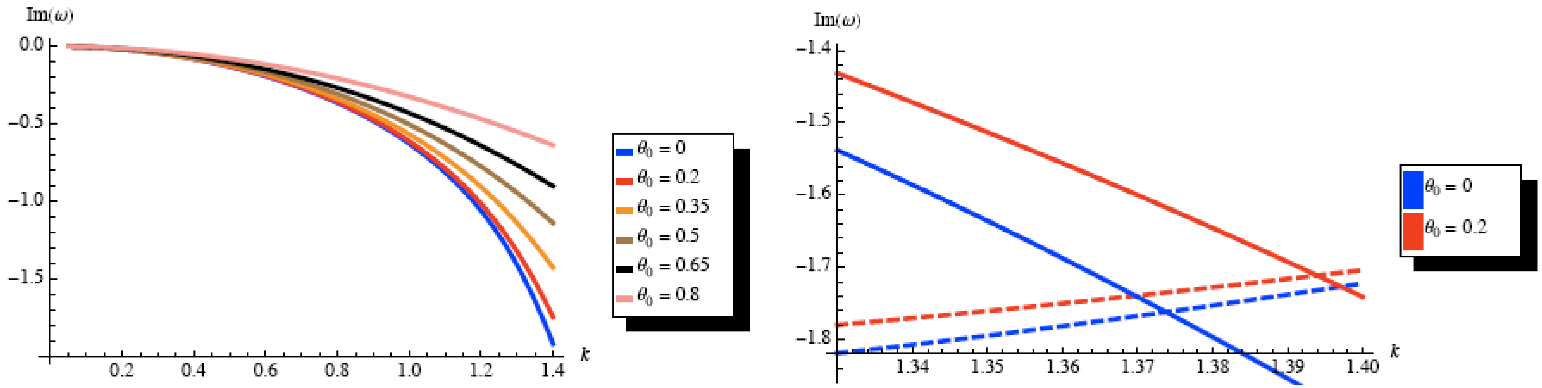
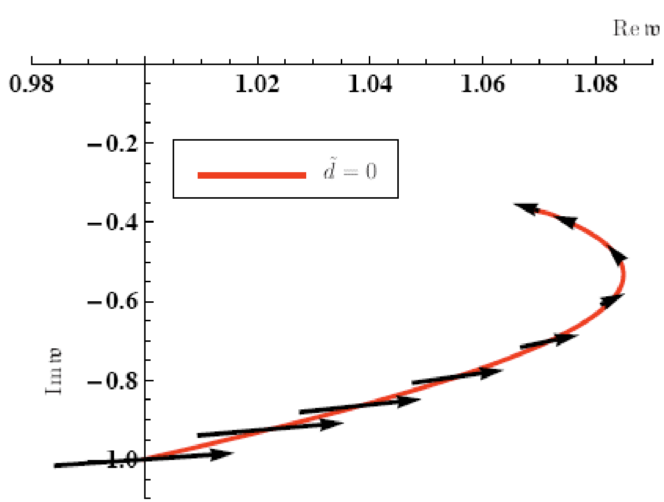


Figure 17: Right: The dispersion relation for the diffusion mode at distinct values for the mass/temperature parameter θ_0. Left: Intersection of the diffusive mode with the imaginary part of the first longitudinal qusinormal mode. The intersection point moves to larger values of k, but to smaller values of Im(ω) as the mass/temperature parameter θ_0 is increased.

Residues



Orientation and magnitude of residues changes considerably when the quark mass is increased (T lowered)!

IV. Conclusion

Summary

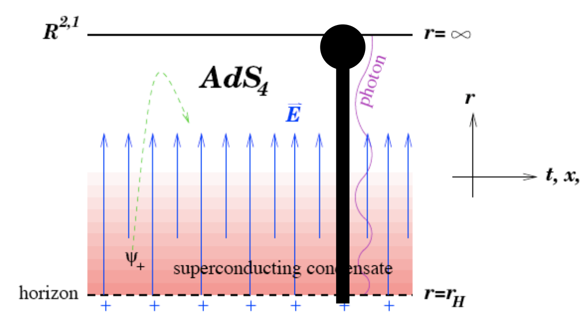
- baryon density 'saves' mesons as QNM/peaks in spec.fun.
- understanding for origin of hydro modes due to SSB (sound)
- determinant formalism for many coupled field fluc's
- phenomenological predictions corrected
- (pseudo)diffusion, fourth sound squared both linear
- fourth sound attenuation finite at Tc

Problems

- what generates the gap?
- spirals & 'attractors'?
- QNMs not complete?

Outlook

- fluctuations in backreacted HHH model
- more scaling, critical exponents
- string setup to get drag in SC phase?
- Fermi surface?



[figure by S.Gubser]

APPENDIX: Details of HHH model

Metric definitions

$$r_H = M^{1/3} L^{2/3}$$

$$T = \frac{3}{4\pi} \frac{r_H}{L^2}$$

$$\begin{pmatrix} r \\ t \\ x \\ y \end{pmatrix} \to \begin{pmatrix} r_H \rho \\ L^2/r_H\, \bar{t} \\ L^2/r_H\, \bar{x} \\ L^2/r_H\, \bar{y} \end{pmatrix} \quad \Rightarrow \quad L^2 \times g,\ M = 1$$

Horizon boundary condition fixed by scalar field current with finite norm

$$J_\mu = \psi^2 A_\mu$$

Physical and dimensionless parameters (chem.pot., charge density, operators)

$$\bar{\mu} = \frac{3L}{4\pi T}\mu \,,$$

$$\bar{n} = \frac{9L}{16\pi^2 T^2} n \,,$$

$$\psi_1 = \frac{3}{4\pi T L^2}\langle O_1 \rangle \,,$$

$$\psi_2 = \frac{9}{16\pi^2 T^2 L^4}\langle O_2 \rangle \,,$$

Horizon asymptotics parametrized by three values

$$\eta = (\rho-1)^\zeta \left(\eta^{(0)} + \eta^{(1)}(\rho-1) + \dots\right) \,,$$

$$\sigma = (\rho-1)^\zeta \left(\sigma^{(0)} + \sigma^{(1)}(\rho-1) + \dots\right) \,,$$

$$a_t = (\rho-1)^{\zeta+1} \left(a_t^{(0)} + a_t^{(1)}(\rho-1) + \dots\right) \,,$$

$$a_x = (\rho-1)^\zeta \left(a_t^{(0)} + a_t^{(1)}(\rho-1) + \dots\right) \,,$$

$$\zeta = -i\omega/3$$

APPENDIX: Details of HHH model 2

Gauge transformations of field fluctuations

$$\delta a_\mu = \partial_\mu \lambda \,,$$
$$\delta\sigma = -\lambda\eta \,,$$
$$\delta\eta = \lambda\sigma + \lambda\psi \,.$$

Green functions

$$G_{\bar{O}_2 O_2} = -\lim_{\Lambda\to\infty}\left(\Lambda^2 \frac{\Psi_q'(\Lambda)}{\Psi_q(\Lambda)} + \Lambda\right) ,$$
$$G_{O_1 \bar{O}_1} = \lim_{\Lambda\to\infty} \frac{\Psi_q(\Lambda)}{\Lambda(\Lambda\Psi_q'(\Lambda) + \Psi_q(\Lambda))} \,.$$

Broken phase determinant $\quad \eta^{IV} = i\lambda\psi \,, \quad \sigma^{IV} = 0 \,, \quad a_t^{IV} = \lambda\omega \,, \quad a_x^{IV} = -\lambda k \,.$

$$0 = \frac{1}{\lambda}\det\begin{pmatrix} {\varphi_\eta}^I & {\varphi_\eta}^{II} & {\varphi_\eta}^{III} & {\varphi_\eta}^{IV} \\ {\varphi_\sigma}^I & {\varphi_\sigma}^{II} & {\varphi_\sigma}^{III} & {\varphi_\sigma}^{IV} \\ \varphi_t^I & \varphi_t^{II} & \varphi_t^{III} & \varphi_t^{IV} \\ \varphi_x^I & \varphi_x^{II} & \varphi_x^{III} & \varphi_x^{IV} \end{pmatrix} \tag{4.15}$$

$$= i\varphi_\eta^{IV}\det\begin{pmatrix} {\varphi_\sigma}^I & {\varphi_\sigma}^{II} & {\varphi_\sigma}^{III} \\ \varphi_t^I & \varphi_t^{II} & \varphi_t^{III} \\ \varphi_x^I & \varphi_x^{II} & \varphi_x^{III} \end{pmatrix} + \omega\det\begin{pmatrix} {\varphi_\eta}^I & {\varphi_\eta}^{II} & {\varphi_\eta}^{III} \\ {\varphi_\sigma}^I & {\varphi_\sigma}^{II} & {\varphi_\sigma}^{III} \\ \varphi_x^I & \varphi_x^{II} & \varphi_x^{III} \end{pmatrix} + k\det\begin{pmatrix} {\varphi_\eta}^I & {\varphi_\eta}^{II} & {\varphi_\eta}^{III} \\ {\varphi_\sigma}^I & {\varphi_\sigma}^{II} & {\varphi_\sigma}^{III} \\ \varphi_t^I & \varphi_t^{II} & \varphi_t^{III} \end{pmatrix} ,$$