International School On

Strings And Fundamental Physics

Munich/Garching, July 25 - August 6, 2010

Introduction to Gauge/Gravity Duality

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Examples Sheet

In order to deepen your knowledge of AdS/CFT, we recommend that you work on at least one exercise for each topic. Note that the exercises for each topic are presented in order of increasing difficulty.

The exercises of this examples sheet are about

Exercise	Topic
I - III	Properties of Anti-de Sitter spacetime
IV - VI	Properties of conformal transformations
VII, VIII	Near-horizon limit of D- and M-branes
IX - XII	Field – Operator matching

Properties of Anti-de Sitter spacetimes

I. Various coordinate systems for AdS_{d+1}.

Lorentzian AdS_{d+1} can be defined by the locus

$$-L^2 = \eta_{ab} X^a X^b = -\left(X^{d+1}\right)^2 - \left(X^0\right)^2 + \sum_{i=1}^{d} \left(X^i\right)^2 \,, \tag{1}$$

where $X \in \mathbb{R}^{2,d}$ and $ds^2 = \eta_{ab} dX^a dX^b$ with $\eta = \mathrm{diag}(-1, 1, 1, \ldots, 1, -1)$. In the following we parametrize the locus (1) in different ways.

a) Draw a picture of AdS_2 embedded in $\mathbb{R}^{2,1}$!

b) The *global coordinates* (ρ, τ, Ω_i) are defined by

$$\begin{aligned} X^{d+1} &= L \cosh\rho \, \sin\tau \,, \\ X^0 &= L \cosh\rho \, \cos\tau \,, \\ X^i &= L \sinh\rho \, \Omega_i \,, \end{aligned}$$

with $i = 1, ..., d$ and $\sum_{i=1}^{d} \Omega_i^2 = 1$. Using this parametrization calculate the induced metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ (where $x^\mu \in \{\rho, \tau, \Omega_i\}$) for AdS_{d+1} in global coordinates.

c) Replace ρ by $r \equiv L \sinh\rho$ and show that the metric can be written in the form

$$ds^2 = -H(r)\,dt^2 + H(r)^{-1}\,dr^2 + r^2\,d\Omega_{d-1}^2,$$

where $d\Omega_{d-1}^2 = \sum_{i=1}^{d} d\Omega_i\,d\Omega_i$ is the metric of the unit $(d-1)$–sphere, S^{d-1}.

d) The *Poincaré patch coordinates* (x^μ, u) with $\mu = 0, ..., d-1$ are defined by

$$\begin{aligned} X^{d+1} + X^d &= u\,, \\ -X^{d+1} + X^d &= v\,, \\ X^\mu &= \frac{u}{L} x^\mu\,. \end{aligned}$$

Use the defining equation (1) to eliminate v in terms of u and x^μ and show that the induced metric for (u, x^μ) with $\mu = 0, ..., d-1$ takes the form

$$ds^2 = L^2 \frac{du^2}{u^2} + \frac{u^2}{L^2} dx^\mu dx_\mu\,.$$

Finally introduce $z = \frac{L^2}{u}$ and show that the metric is given by

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + dx^\mu dx_\mu\right)\,.$$

Which part of the AdS spacetime is not covered by these coordinates? (Hint: z takes only positive values (Why?))

II. Geodesics in global AdS

We consider *global AdS* given by the coordinates (ρ, τ, Ω_i) and the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = L^2\left(-\cosh^2\rho\; d\tau^2 + d\rho^2 + \sinh^2\rho\; d\Omega_i^2\right)\,.$$

a) What is the condition satisfied by ds^2 for a radially-directed light-ray? Using this and starting from $\rho = \rho_0$ with proper time $\tau(\rho_0) = 0$, find the trajectory $\tau(\rho)$ for such a light-ray. What is the coordinate time for a geodesic to go from ρ_0 to the boundary and come back? What is the proper time measured by a stationary observer's clock at ρ_0 for this trajectory? Comment on this!

b) What is the behaviour of a massive geodesic in the radial direction of AdS? Show that a massive geodesic never reaches the conformal boundary of AdS.

Hints for exercise b)

(i) The norm of the velocity $u^\mu = \frac{dx^\mu}{dT}$ is always -1, where T is the proper time along the worldline.

(ii) We know that $\cosh^2\rho\,\frac{d\tau}{dT} = C = \text{const.}$ (Why?)

III. Poincaré patch of AdS

Calculate the inverse metric $g^{\mu\nu}$, the Christoffel symbols $\Gamma^\mu_{\rho\sigma}$ as well as the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ in the Poincaré patch of AdS_{d+1}. Confirm that this patch is a solution of Einstein's field equations

$$G_{\mu\nu} - \Lambda g_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} - \Lambda g_{\mu\nu} = 0$$

and determine the value of the cosmological constant Λ.

<u>Hints:</u> The Christoffel symbols $\Gamma^\mu_{\rho\sigma}$ and the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ are given by

$$\begin{aligned}
\Gamma^\mu_{\rho\sigma} &= \frac{1}{2} g^{\mu\nu} \left(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma} \right) , \\
R^\mu{}_{\nu\rho\sigma} &= \partial_\rho \Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\nu\sigma} - (\rho \leftrightarrow \sigma) .
\end{aligned}$$

The Ricci tensor and Ricci scalar are determined by $R_{\nu\sigma} = R^\mu{}_{\nu\mu\sigma}$ and $R = R^\nu{}_\nu$.

Properties of conformal transformations

IV. Conformal Symmetry

a) Show that a special conformal transformation may be written as a combination of an inversion, a translation and another inversion.

b) Consider a curved space with a metric of Euclidean signature. An infinitesimal *Weyl transformation* of the metric is given by

$$\delta g^{\mu\nu}(x) = 2\sigma(x) g^{\mu\nu}(x) ,$$

with a scalar function $\sigma(x)$. Show that the conformal Killing equation may be obtained by combining an infinitesimal diffeomorphism with an infinitesimal Weyl transformation, and subsequently reducing to flat space.

c) Show that the two-point function of the scalar field $\mathcal{O}$ with conformal dimension Δ on a flat Euclidean d-dimensional space,

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \frac{C}{(x-y)^{2\Delta}}$$

is covariant under conformal transformations.

V. Conformal Algebra

Show that the conformal algebra of the spacetime $\mathbb{R}^{q,p}$, given by the generators $M_{\mu\nu}$, P_μ, K_μ and D, is isomorphic to SO(q+1, p+1).

The commutators read:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= i \left(\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} \right) , \\
[K_\rho, M_{\mu\nu}] &= i \left(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu \right) , \\
[P_\rho, M_{\mu\nu}] &= i \left(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu \right) , \\
[D, P_\mu] &= i P_\mu , \quad [D, K_\mu] = -i K_\mu , \\
[K_\mu, P_\nu] &= 2i \left(\eta_{\mu\nu} D - M_{\mu\nu} \right) ,
\end{aligned}$$

where $\eta_{\mu\nu}$ are the entries of a diagonal matrix with q times -1 and p times 1 on the diagonal. Conclude that the conformal algebra of $\mathbb{R}^{1,p}$ is the isometry group of AdS_{p+2}.

VI. Infinitesimal conformal transformations

Show that the generator of an infinitesimal version of the conformal transformations in more than two spacetime dimensions is at most second order in the spacetime coordinates.

Hint: Use the definition $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\rho \epsilon^\rho$ and show that the third derivatives of ϵ have to vanish.

Near-horizon limit of D- and M-branes

VII. Near-horizon limit of D3-branes

The near-horizon limit of the metric of D3-branes was already discussed in the lecture. Determine in this exercise the near-horizon limit of the Ramond-Ramond five-form field strength tensor $F_{(5)}$, which is given by

$$F_{(5)} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dH^{-1} + \dots \,,$$

where the terms denoted by dots are choosen such that $F_{(5)}$ is self-dual. The function $H(r)$ is given by

$$H(r) = 1 + \frac{L^4}{r^4} \,.$$

VIII. Near-horizon limit of M2-branes

The supergravity solution of N coincident M2–branes reads

$$\begin{aligned} ds^2 &= H(r)^{-2/3}\left(-dt^2 + dx^2 + dy^2\right) + H(r)^{1/3}\left(dr^2 + r^2 d\Omega_7\right) \,, \\ F_{(4)} &= dt \wedge dx \wedge dy \wedge dH^{-1} \,, \end{aligned}$$

where $H(r)$ is given by

$$H(r) = 1 + \frac{L^6}{r^6} \,, \quad \text{where} \quad L^6 = 32\pi^2 N l_p^6.$$

a) Take the near-horizon limit $r \to 0$ and calculate the metric and the four-form $F_{(4)}$ in this limit.

b) Use the coordinate transformation $z = \frac{L^3}{2r^2}$ and compute the metric as well as the four-form $F_{(4)}$ in the coordinates (z, t, x, y, Ω_7). Which is the spacetime you obtain in this way?

Field – Operator matching

IX. Massive scalar fields in AdS

Let us consider a massive scalar field in Euclidean AdS in the Poincaré patch.

a) Derive the equations of motion for a scalar field with mass m in Euclidean AdS_{d+1}.

b) Use the ansatz $\phi(z) = z^\Delta$ near the boundary $z \to 0$ and determine the two possible values of $\Delta_\pm$, where $\Delta_+ > \Delta_-$.

c) What is the scaling dimension of the corresponding scalar operator $\mathcal{O}$ on the field theory side?

X. Bulk Gauge Fields in AdS/CFT

Let us consider massive gauge fields in AdS_{d+1} by the Proca action

$$S = \int_{AdS} d^{d+1}x\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu\right) .$$

a) Derive the equations of motion for A_μ in the Poincaré patch of Euclidean AdS_{d+1}!

b) Determine the index Δ by plugging the ansatz $A_\mu(z) = z^\Delta$ into the equations of motions!

c) What is the scaling dimension of the corresponding current on the field theory side, which is dual to the bulk gauge field? The coupling of the current to the bulk field is given by

$$\int_{\partial AdS_{d+1}} d^dx\sqrt{\gamma}A_\mu J^\mu ,$$

where $\gamma_{\mu\nu}$ is the induced metric on the conformal boundary of AdS_{d+1}.

XI. Saturation of unitarity bound

Consider a free scalar field with mass m in the Poincaré patch of Euclidean AdS_{d+1}.

a) Show that the usual bulk action

$$S_1 = -\frac{1}{2}\int d^dx\, dz\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) ,$$

evaluated for solutions of the form $\phi(x,z) \sim z^\Delta e^{ikx}$ near $z = 0$, is finite if $\Delta > d/2$, i.e. the solution with $\Delta = \Delta_+$ (Δ_+ being the larger root of $\Delta(\Delta - d) = m^2L^2$) are normalizable with respect to the action S_1.

b) Consider the bulk action

$$S_2 = -\frac{1}{2}\int d^dx\, dz\sqrt{g}\phi\left(-\Box_g + m^2\right)\phi .$$

Show that S_2 can be obtained by adding a boundary term to the action S_1 (hint: partial integration) and that the equations of motion are the same as for the action S_1.

c) Show, that the action S_2 evaluated for solutions of the form $\phi(x,z) \sim z^\Delta e^{ikx}$ near $z = 0$ is finite if $\Delta > (d-2)/2$. Conclude that for

$$-\frac{d^2}{4} < m^2L^2 < -\frac{d^2}{4} + 1 ,$$

both solutions, i.e. $\Delta = \Delta_+$ and $\Delta = \Delta_-$ are normalizable with respect to the action S_2.

XII. Propagators

Let us consider a massive scalar field in Euclidean AdS in the Poincaré patch (see exercise IX).

The bulk-to-boundary propagator $K(x, z; x')$ is the solution of the equations of motion which is regular in the interior and diverges like

$$\lim_{z \to \epsilon} K(z, x; x') = \epsilon^{\Delta_-} \delta(x - x')$$

near the boundary, i.e. for $\epsilon \ll 1$. The bulk-to-bulk propagator is given by the solution of the equation of motion with a pointlike source term,

$$\left(\Box_{x,z} - m^2\right) G(z, x; z', x') = \frac{1}{\sqrt{g}} \delta(x - x')\delta(z - z'),$$

where $\Box_{x,z}$ is the scalar Laplacian in Euclidean AdS_{d+1} acting only on x and z. Moreover the bulk-to-bulk propagator is regular in the interior.

Show that the bulk-to-boundary propagator $K(x, z; x')$ can be calculated from the bulk-to-bulk propagator $G(z, x; z', x')$ by

$$K(z, x; x') = \lim_{z' \to \epsilon} \frac{\Delta_+ - \Delta_-}{\epsilon^{\Delta_+}} G(z, x; z', x').$$

Hint: Do not use the explicit solution for $G(z, x; z', x')$, but Green's second identity in the following way

$$\int_M d^d x dz \sqrt{g} \left[\phi(\Box_{x,z} - m^2)\psi - \psi(\Box_{x,z} - m^2)\phi\right] = \int_{\partial M} d^d x \sqrt{\gamma}(\phi \partial_n \psi - \psi \partial_n \phi),$$

where γ is the determinant of the induced metric on ∂M and ∂_n is the derivative normal to the boundary, i.e. in our case $\partial_n = \partial_z$.