Conformal Bootstrap: Problems for Lectures 1 and 2

1. The representation of the generators of the conformal algebra in d (Euclidean) dimensions in terms of differential operators reads

$$\begin{aligned} P_\mu &= i\partial_\mu && (1)\\ M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) && (2)\\ D &= ix^\mu\partial_\mu && (3)\\ K_\mu &= i(2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu) && (4)\end{aligned}$$

Compute their commutators. Verify that the algebra is isomorphic to $SO(d+1,1)$, under the identification

$$J_{\mu\nu} = M_{\mu\nu}\,, \quad J_{\mu+} = P_\mu \quad J_{\mu-} = K_\mu \quad J_{+-} = D\,. \tag{5}$$

Here we have introduced lightcone coordinates in $\mathbb{R}^{d+1,1}$,

$$X^\pm = X^{d+2} \pm X^{d+1}\,. \tag{6}$$

2. Consider the special case of two-dimensional conformal field theory. Recall the Virasoro generators

$$L_n = -z^{n+1}\frac{\partial}{\partial z}\,, \quad \bar{L}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}\,, \tag{7}$$

where $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ are complex coordinates on the plane. Find the relation between the conformal generators as presented in class (P_μ, K_μ, $M_{\mu\nu}$ and D, where $\mu,\nu = 1,2$) and the generators of the "global" part of the Virasoro algebra (also known as the Moebius algebra) namely L_n, $\bar{L}_n$ with $n = -1, 0, 1$.

Show that the the exponentiation of the Moebius algebra gives the group $SL(2,\mathbb{C})$ of fractional linear transformations,

$$z' = \frac{az+b}{cz+d}\,, \quad ad - bc = 1\,. \tag{8}$$

3. Constrain the form of 2-, 3- and 4-point functions of conformal *scalar* operators in $\mathbb{R}^d$ (d-dimensional Euclidean space) of a priori different dimensions. Here are two equivalent ways to do this:

- Using the embedding in $\mathbb{R}^{d+1,1}$ explained in the lectures.
  Recall the definition of the projective hypercone in $\mathbb{R}^{d+1,1}$,

$$\eta_{MN} X^M X^N = 0\,, \quad M, N = 1, \ldots d+2\,, \quad X^M \sim \lambda X^M \tag{9}$$

  where η has signature $(+ + + \cdots -)$. The embedding of $\mathbb{R}^d$ into the hypercone is

$$x_\mu = \frac{X_\mu}{X_{d+2} + X_{d+1}}\,, \quad \mu = 1, \ldots d\,. \tag{10}$$

  We choose a section of the hypercone by imposing

$$X^+ \equiv X_{d+2} + X_{d+1} = 1\,. \tag{11}$$

  A useful elementary property is

$$X_i \cdot X_j = -\frac{1}{2} x_{ij}^2\,, \quad x_{ij}^2 \equiv (x_i - x_j)^2\,. \tag{12}$$

  In this language, a scalar operator $\mathcal{O}(x)$ of dimension Δ corresponds to a scalar operator $\mathcal{O}^{HC}(X)$ defined on the hypercone, according to

$$\mathcal{O}(x) = (X^+)^\Delta \mathcal{O}^{HC}(X) = \mathcal{O}^{HC}(X)\,. \tag{13}$$

  The hypercone operators obeys

$$\mathcal{O}^{HC}(\lambda X) = \lambda^{-\Delta} \mathcal{O}(X)\,. \tag{14}$$

- An alternative (perhaps more elementary) way to constrain conformal correlators is to use covariance under inversion,

$$I\,: x_\mu \to x'_\mu = \frac{x_\mu}{x^2}\,. \tag{15}$$

  A little calculation gives

$$x'^2_{12} = \frac{x_{12}^2}{x_1^2 x_2^2}\,. \tag{16}$$

  Show that this leads to the same conclusions as the embedding method.

Repeat the exercise for the 2-point function of vector operators $\mathcal{O}_\mu(x)$ of dimension Δ. The correspond to vector operators $\mathcal{O}_M(x)$ on the hypercone obeying

$$\mathcal{O}_M^{HC}(\lambda X) = \lambda^{-\Delta}\mathcal{O}_M^{HC}(X)\,, \quad X^M\mathcal{O}_M^{HC}(X) = 0\,, \quad \mathcal{O}_M^{HC}(X) \sim X_M\mathcal{O}^{HC}(X)\,. \tag{17}$$

where the last relation should be interpreted as a gauge equivalence. The $4d$ operator is recovered by projecting the index as follows,

$$\mathcal{O}_\mu(x) = \frac{\partial X^M}{\partial x^\mu}\mathcal{O}_M^{HC}(X)\,. \tag{18}$$

4. Give as many concrete examples as you can for the different types of non-generic representations of the four-dimensional conformal group introduced in class: $\mathcal{C}_{j_1,j_2}$, $\mathcal{B}^L_{j_1}$, $\mathcal{B}^R_{j_2}$ and $\mathcal{B}$. For example, we saw that $\mathcal{B}$ corresponds to a free scalar field, obeying $\Box\phi = 0$.

5. The character of a representation R of the four-dimensional conformal group is defined as
$$\chi_R = \mathrm{Tr}_R\, s^{2\Delta}x^{2j_1}\bar{x}^{2j_2}\,. \tag{19}$$
Recall that the decomposition rules of conformal representations at the unitarity bounds are:

$$\begin{aligned}
\lim_{\epsilon\to 0}\mathcal{A}_{j_1+j_2+2+\epsilon,j_1,j_2} &= \mathcal{C}_{j_1,j_2} + \mathcal{A}_{j_1+j_2+3,j_1-\frac{1}{2},j_2-\frac{1}{2}} && (20)\\
\lim_{\epsilon\to 0}\mathcal{A}_{j_1+1+\epsilon,j_1,0} &= \mathcal{B}^L_{j_1} + \mathcal{C}_{j_1-\frac{1}{2},\frac{1}{2}} && (21)\\
\lim_{\epsilon\to 0}\mathcal{A}_{j_2+1+\epsilon,0,j_2} &= \mathcal{B}^R_{j_2} + \mathcal{C}_{\frac{1}{2},j_2-\frac{1}{2}} && (22)\\
\lim_{\epsilon\to 0}\mathcal{A}_{1+\epsilon,0,0} &= \mathcal{B} + \mathcal{A}_{3,0,0}\,. && (23)
\end{aligned}$$

Compute $\chi_{\mathcal{C}_{j_1,j_2}}$, $\chi_{\mathcal{B}^L_{j_1}}$, $\chi_{\mathcal{B}^R_{j_2}}$ and $\chi_{\mathcal{B}}$.

Challenging problem:

Using characters, find the decomposition into irreducible representations of the tensor product $\mathcal{B}\otimes\mathcal{B}$. This amounts to decomposing into irreps the most general bilinear of a free complex scalar field,

$$:\partial_{\mu_1}\ldots\partial_{\mu_k}\phi\,\partial_{\nu_1}\ldots\partial_{\nu_\ell}\bar{\phi}:\;. \tag{24}$$

(Why complex and not real?).

Hints: guess the answer by doing a few steps of the *sieve* procedure or by thinking about the field theory interpretation. Use Mathematica to handle the characters.