Spectral Theory of Inductive Limit C^*-Algebras and Application to Loop Quantum Gravity

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Abstract:

Projective limits of topological spaces are a well known tool for the construction of quantum configuration spaces in loop quantum gravity, where the latter appear as the spectra of certain C^*-algebras. In this thesis this construction is embedded in a general theory. It is shown, that expressing a quantum configuration space as a projective limit corresponds always dually to an inductive limit construction on the corresponding C^*-algebra. Afterwards it is investigated in the case of C^*-dynamical systems, which construction on the C^*-side corresponds to a quotient of the quantum configuration space by a group action and moreover the compatibility of those notions with inductive and projective limits is investigated. Further the representation theory of inductive limit C^*-algebras is analyzed and it is shown that cyclic representations of an inductive limit C^*-algebra arise precisely as an inductive limit of the GNS representations of the members of the inductive family. Finally those concepts are applied in the theories of polymer quantization and loop quantum gravity, where the corresponding algebras of configuration variables are expressed as inductive limits of more elementary C^*-algebras. Furthermore the quantum configuration spaces are calculated using the spectra of the elementary C^*-algebras as well as the methods presented earlier in this thesis.

Selbständigkeitserklärung

Ich erkläre hiermit, dass ich die vorliegende Arbeit selbständig verfasst und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe. Weiterhin erkläre ich, eine Masterarbeit in diesem Studiengebiet erstmalig einzureichen.

München, den 20.8.2018 ....

Statement of authorship

I declare that I completed this thesis on my own and that information which has been directly or indirectly taken from other sources has been noted as such. Neither this nor a similar work has been presented to an examination committee before.

Munich, August 20, 2018 ....

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1. Introduction

In mathematics it is a typical approach, to define mathematical objects using easier objects involved in some construction. Elementary examples for this procedure are for example quotient spaces, where complicated spaces arise as quotients of more elementary spaces by non trivial equivalence relations or manifolds, which are defined by patching together euclidean spaces. So called inductive and projective limits follow the same philosophy: Inductive systems are rather large commutative diagrams consisting out of "easy" objects and embeddings of the objects into each other. The inductive limit of such a diagram is now an object, which could be understood as a minimal object in which the whole diagram can be embedded. Dually, a projective system consists out of objects and projections and its projective limit is in some sense the minimal object that "projects" on the diagram. The defining property of inductive and projective systems, which gives them a special role in the zoo of colimits and limits, is, that the objects in the diagram are associated to the elements of a directed, partially ordered set and that the occuring maps have to satisfy some compatibility relations according to the order structure of the partially ordered set. In this thesis we consider the case of inductive limits of C^*-algebras, i.e. we consider inductive limits of "simple" C^*-algebras to obtain more complicated ones.

It is now a typical question in mathematics, to which extent properties of the simple objects involved in a construction mirror themselves in properties of the complicated object obtained by the construction. In the example of quotient spaces one could ask for example, in which situations the quotient space is Hausdorff, given that the total space is Hausdorff. In this thesis we focus on spectral properties of abelian, unital C^*-algebras and want to understand, to which extent spectral properties of the "simple constituents" determine spectral properties of the "complicated" inductive limit C^*-algebra.

At this stage it is convenient to explain more accurately what is meant by "spectral properties of a C^*-algebra". Therefore recall first a well known formulation of the spectral theorem from functional analysis:

Theorem 1 (cp. Thm. 29.2.1 of [31])
Let A be a bounded, normal operator on a Hilbert space $\mathcal{H}$. Let $\sigma(A)$ be the spectrum of A. Then there exists a unitary operator

$$U : \mathcal{H} \to \bigoplus_{i \in I} L^2(\sigma(A), d\mu_i)$$

where $d\mu_i$ are regular Borel measures on $\sigma(A)$ such that UaU^{-1} becomes the multiplication operator $\lambda\cdot$ on each subspace $L^2(\sigma(A), d\mu_i)$.

This gives a different characterization of the spectrum of an operator: On the one hand it can be seen as a generalized concept for eigenvalues. On the other hand it can be seen as the domain of a L^2 space, on which the operator can be represented by a multiplication operator. This motivates the following intuitive meaning of the word "spectrum" in the context of abelian, unital C^*-algebras:

> Intuition: *The spectrum of an abelian, unital C^*-algebra $\mathfrak{A}$ is a topological space $\Delta(\mathfrak{A})$, such that $\mathfrak{A}$ can be represented by multiplication operators on $L^2(\Delta(\mathfrak{A}), d\mu)$ for some measure $d\mu$. I.e.: For each $a \in \mathfrak{A}$ there exists a function $\check{a} \in C(\Delta(\mathfrak{A}))$ such that the map $\pi : a \mapsto \check{a}(\lambda)\cdot$ is a representation for $\mathfrak{A}$.*

In the second chapter the spectrum of a C^*-algebra will be defined in a different way. But we will show the statement of above intuitive definition by showing a so called spectral theorem for abelian, unital C^*-algebras, which states, that any abelian, unital C^*-algebra can be represented by multiplication operators on a L^2 space over its spectrum. After this short discussion one should have understood, what is meant by spectral properties of a C^*-algebra $\mathfrak{A}$: Herewith the structure of the spectrum $\Delta(\mathfrak{A})$ and the structure of the representation by multiplication operators over the spectrum is meant. Hence the first basic question of this thesis is:

> Question 1: *How are the spectra of an inductive family of C^*-algebras related to the spectrum of the corresponding inductive limit C^*-algebra and how are the corresponding representations by multiplication operators related to each other?*

At this stage it is convenient to recall, that C^*-algebras arise in quantum field theory as algebras of observables. A representation of such an algebra of observables gives then an algebra of operators on a Hilbert space. For example, this can be used to calculate expectation values, cross sections, or other observable quantities. Now one of the cornerstones of modern physics is the gauge principle, which in this situation incorporates the statement, that a postulated set of observables is often overcomplete in the sense, that it distinguishes physically indistinguishable states. Further the gauge principle claims, that this redundancy is mediated by a gauge group in the sense, that the physical configuration space is given by the quotient of some "overcomplete" space by a gauge group action. Dually this means, that physically realizable observables modelled on the overcomplete space have to be invariant under the gauge group action. This example motivates the introduction of group actions on C^*-algebras, which will be denoted by C^*-dynamical systems. In this case, the algebra of invariant observables corresponds to the so called fixed point algebra of a C^*-dynamical system. This motivates the second basic question of this thesis:

> Question 2: *How is the spectrum of the fixed point subalgebra (i.e. the spectrum of the algebra of invariant elements) of a C^*-algebra related to the spectrum of the full C^*-algebra?*

As explained above, we use inductive limits of C^*-algebras to construct complicated C^*-algebras using more elementary ones. Further we wish at this point, that this construction is also a useful one in the case of C^*-dynamical systems. Namely, that a C^*-dynamical system can be defined as an inductive limit of C^*-dynamical systems and moreover that the fixed point algebra of the latter can be calculated using only the fixed point algebras of the former. This gives the third big question of this thesis:

> Question 3: *How do C^*-dynamical systems behave under inductive limits and is it possible to calculate the fixed point subalgebra of an inductive limit C^*-algebra using only the fixed point subalgebras of its constituents?*

If the answers of above questions would give, that the spectral and fixed point properties of inductive limit C^*-algebras are determined in a convenient way totally by the corresponding properties of their constituent C^*-algebras and that the spectrum of the fixed point algebra is determined in a convenient way by the spectrum of the full algebra, then we would have a powerful machinery to calculate spectra of complicated C^*-algebras which arise as inductive limits. We will see later on, that the situation becomes even more convenient, since the construction which relates the spectrum of an inductive limit C^*-algebra to the spectra of its constituents is much better controllable than the inductive limit construction in the case of C^*-algebras itself. The reason for this is, that the latter incorporates a completion, which is always a reason for bad feelings.

After reading so much abstract nonsense, the impatient physicist may asks himself: What is that good for? To clarify this question, we want to give a perspective on the physical meaning of the spectrum of a C^*-algebra. Therefore consider the case of quantum mechanics with a single degree of freedom. The classical phase space $\mathcal{P}$ of this system is coordinatized by two canonical coordinates $p, q : \mathcal{P} \to \mathbb{R}$ which satisfy the following Poisson bracket relations:

$$\begin{aligned} \{q, p\} &= 1 \\ \{p, p\} = \{q, q\} &= 0 \end{aligned}$$

By the postulate of canonical quantization, we now want to find a representation of p, q by operators $\hat{p}, \hat{q}$ on a Hilbert space $\mathcal{H}$, satisfying the canonical commutation relations:

$$\begin{aligned} [\hat{q}, \hat{p}] &= i\hbar \\ [\hat{p}, \hat{p}] = [\hat{q}, \hat{q}] &= 0 \end{aligned}$$

One such representation is given by the Schrödinger representation, where the Hilbert space is choosen to be $\mathcal{H} = L^2(\mathbb{R}, d\lambda)$ and the operators are represented by:

$$\begin{aligned} \hat{q} &= \lambda \cdot \\ \hat{p} &= -i\hbar \frac{d}{d\lambda} \end{aligned}$$

Especially we see, that $\hat{q}$ is represented by the multiplication operator $\lambda\cdot$. In this situation $\mathbb{R}$ can be called the quantum configuration space of the system, since square integrable functions over $\mathbb{R}$ give the quantum states of the system. By comparing this situation with above intuitive definition for the spectrum of a C^*-algebra, one obtains, that the spectrum of the algebra of configuration variables can be understood as the quantum configuration space over which the quantum system is modelled as a L^2 space together with a representation of the C^*-algebra of configuration variables by multiplication operators. Further one can show, that this special representation is in the sense universal, that any representation of this algebra is a direct sum of such representations. Hence the spectrum of a C^*-algebra of observables determines exactly the quantum configuration space whose corresponding states are all distinguishable by the postulated algebra of physical configuration variables without being overcomplete. This fact can be illustrated in the case of gauge theories: The spectrum of the full C^*-algebra of configuration variables is in general larger than the spectrum of the fixed point subalgebra, since the first is the quantum configuration space whose corresponding states can be distinguished by measurements corresponding to all observables, while the latter determines the states which can be distinguished by measurements of gauge invariant observables only.

Hence the methods developed in this thesis can be useful to compute quantum configuration spaces of quantum systems, whose algebras of configuration variables are expressible as inductive limit C^*-algebras. Before talking about physical theories where such constructions are indeed useful, we want to give the following "algorithm" for the computation of quantum configuration spaces of such physical systems, which is motivated by the methods of this thesis (Please note, that the choices made in 1. and 2. are far from being unique: Different choices of polarizations correspond to different abelian, unital subalgebras of the full quantum algebra and hence give in general different spectra!):

1. Choose a set of classical configuration variables with vanishing Poisson bracket (i.e. choose a polarization of the phase space, cp. [32]).
2. Define an abelian, unital C^*-algebra of quantum configuration variables using this classical Poisson algebra.
3. Express the C^*-algebra as an inductive limit of more elementary C^*-algebras.
4. Calculate the spectra of the constituents.
5. Calculate the spectrum of the full algebra using the methods developed in this thesis.

Of course this algorithm would not be very useful, if there was no physical system whose quantum algebra is expressible as an inductive limit C^*-algebra. But fortunately this is the case in loop quantum gravity and related theories. Here projective techniques are a well known tool for the construction of quantum configuration spaces and used from the 90s on (cp. [6], [5], [33]). But somehow, a coherent theory, if and how this projective limits of topological spaces are related to constructions on the C^*-side, was lacking. This situation was the original motivation to deal with the subjects presented here. Further the final purpose of this thesis is to fill this gap and illustrate the relevance of the investigated concepts in the case of loop quantum gravity.

We now want to give an overview over the content of this thesis. The first part deals with the spectral theory of inductive limit C^*-algebras and C^*-dynamical systems. Therefore in the second section fundamental notions regarding the theory of operator algebras are introduced. The "first question" from above is answered in the third section. Therefore inductive and projective limits are introduced and applied on Banach spaces, compact Hausdorff spaces, Hilbert spaces, C^*-algebras, algebras of continuous functions and measure spaces. Further the behaviour of important concepts in C^*-theory, as spectra, states, representations and the Gel'fand transform, under inductive limits and projective limits is analyzed. Finally a structure theorem (the above mentioned spectral theorem) for cyclic representations is given. The fourth section deals with the above mentioned "second question" and "third question". Therefore group actions on compact Hausdorff spaces as well as C^*-dynamical systems are introduced and it is shown, that the spectrum of the fixed point algebra is given by the quotient of the spectrum of the full algebra by some induced group action. Afterwards the compatibility of those concepts with inductive and projective limits is analyzed. In the second part of this thesis the general theory developed in first part is applied to the cases of polymer quantization of the real scalar field and loop quantization of gravity. The former is a quantization procedure for scalar fields, which mimicks important features of full loop quantum gravity and could be regarded as a toy example for loop quantization. In both sections the quantum configuration spaces are calculated by following the above stated algorithm, which gives - among other things - an inductive limit decomposition of the corresponding quantum algebras. In the last section the results of this thesis are discussed and possible directions for further research are presented.

Part I.

Spectral Theory of Inductive Limit C^*-Algebras

2. A Crash Course in Operator Algebras

This chapter provides a brief introduction to the theory of operator algebras. In the first section the most important types of operator algebras, including Banach- and C^*-algebras, are introduced and elementary facts are presented for later use. In the second section the notion of the spectrum of an operator algebra is introduced and related notions are presented. In the third section the important case of abelian, unital C^*-algebras is insvestigated and its connection to the algebra of continuous functions on compact Hausdorff spaces is discussed. In the next two sections the notions of states and representations are introduced, yielding the necessary background for the introduction of the GNS-construction, which is presented in the following section. Finally a spectral theorem for C^*-algebras is shown, which gives a different characterization of the GNS-representation in terms of multiplication operators and L^2 spaces. The main references for this section are [31], [19], [10]. Except of the last section, this section contains almost no proofs, since the introduced concepts and objects are elementary in the theory of C^*-algebras and can be found in almost any book on operator algebras.

2.1. Basic Definitions and Basic Facts

First we want to define the very basic notions related to operator algebras:

Definition 1 (Basic notions, cp. [19], [10])
Let $\mathfrak{A}$ be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then:

1. *An algebra is a tuple $(\mathfrak{A}, \cdot)$ where $\mathfrak{A}$ is a vector space and $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}, (a, b) \mapsto a \cdot b =: ab$ is a multiplication map for which the following holds:*
   a) $\forall a, b, c \in \mathfrak{A} : (ab)c = a(bc)$ *(Associativity)*
   b) $\forall a_1, a_2, b \in \mathfrak{A}, \forall \alpha, \beta \in \mathbb{K} : b(\alpha_1 a_1 + \alpha_2 a_2) = \alpha_1 b a_1 + \alpha_2 b a_2$
   c) $\forall a_1, a_2, b \in \mathfrak{A}, \forall \alpha, \beta \in \mathbb{K} : (\alpha_1 a_1 + \alpha_2 a_2)b = \alpha_1 a_1 b + \alpha_2 a_2 b$
2. *An algebra $(\mathfrak{A}, \cdot)$ is called abelian, iff $\forall a, b \in \mathfrak{A} : [a, b] = ab - ba = 0$.*
3. *An algebra is called unital, iff there is a $\mathbb{1} \in \mathfrak{A}$ with $\forall a \in \mathfrak{A} : \mathbb{1}a = a\mathbb{1} = a$. In this case we often denote the identity also by $\mathbb{1}_{\mathfrak{A}}$.*
4. *A linear subspace $\mathfrak{B} \subset \mathfrak{A}$ is called a subalgebra, iff $\forall a, b \in \mathfrak{B} : ab \in \mathfrak{B}$.*
5. *A subalgebra $\mathfrak{J} \subset \mathfrak{A}$ is called a left (right) ideal, iff $\forall a \in \mathfrak{A}, b \in \mathfrak{J} : ab \in \mathfrak{J}$ ($ba \in \mathfrak{J}$). An ideal which is a left and a right ideal is called a two-sided ideal.*
6. *An ideal (of either kind) is called maximal, iff there is no other ideal containing it except for $\mathfrak{A}$ itself.*
7. *An involution on an complex algebra $\mathfrak{A}$ is a map $* : \mathfrak{A} \to \mathfrak{A}, a \mapsto a^*$ satisfying:*
   a) $\forall a, b \in \mathfrak{A}, \alpha, \beta \in \mathbb{C} : (\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$ *(antilinear)*

b) $\forall a, b \in \mathfrak{A} : (ab)^* = b^* a^*$.

c) $\forall a \in \mathfrak{A} : (a^*)^* = a$

8. *An algebra/subalgebra/ideal which is equipped with an involution is called a $*$-algebra/$*$-subalgebra/$*$-ideal.*

9. *A morphism ($*$-morphism) is a linear map $\phi : \mathfrak{A} \to \mathfrak{B}$ between algebras ($*$-algebras) which preserves the multiplicative (and involutive) structure, i.e. $\forall a, b \in \mathfrak{A} : \phi(a)\phi(b) = ab$ (and $\phi(a^*) = \phi(a)^*$).*

10. *An isomorphism ($*$-isomorphism) is a bijective morphism ($*$-morphism).*

11. *An unital morphism ($*$-morphism) is a morphism ($*$-morphism) which preserves the unit-element.*

12. *A normed algebra $\mathfrak{A}$ is an algebra together with a norm $\|\cdot\| : \mathfrak{A} \to \mathbb{R}^+$, such that $\forall a, b \in \mathfrak{A} : \|ab\| \leq \|a\|\|b\|$. If $\mathfrak{A}$ is a $*$-algebra, we further demand $\forall a \in \mathfrak{A} : \|a^*\| = \|a\|$. If it is an unital algebra, we further demand $\|\mathbb{1}_{\mathfrak{A}}\| = 1$.*

13. *A normed algebra ($*$-algebra) is called a Banach algebra (Banach $*$-algebra), if it is complete as a normed space.*

14. *A C^*-algebra is a Banach $*$-algebra $\mathfrak{A}$ such that the C^*-property $\forall a \in \mathfrak{A} : \|a^* a\| = \|a\|^2$ holds.*

We further have the following elementary Lemma:

Lemma 1 ($*$-morphisms are norm decreasing, cp. Cor. II.1.6.6 of [9])
Let $\mathfrak{A}_1, \mathfrak{A}_2$ be C^*-algebras with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively and let $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ be a $*$-morphism. Then: ϕ is norm decreasing, i.e.

$$\forall a \in \mathfrak{A}_1 : \|\phi(a)\|_2 \leq \|a\|_1$$

and hence also bounded.

We now want to define the notion of a quotient algebra

Definition 2 (Quotient algebra, cp. Sec. 2.1.1 of [10])
Let $\mathfrak{A}$ be C^*-algebra and $\mathfrak{I} \subset \mathfrak{A}$ be a closed, 2-sided ideal. Then define the quotient algebra of $\mathfrak{A}$ by $\mathfrak{I}$ to be the linear space $\mathfrak{A}/\mathfrak{I}$ together with the norm

$$\forall [a] \in \mathfrak{A}/\mathfrak{I} : \|[a]\|_{\mathfrak{A}/\mathfrak{I}} = \inf_{b \in \mathfrak{I}} \|a + b\|_{\mathfrak{A}}$$

and the $*$-algebra structure:

$$\forall a, b \in \mathfrak{C} : [a] \cdot [b] = [ab]$$
$$\forall a \in \mathfrak{C} : [a]^* = [a^*]$$

We then have:

Lemma 2 (Well definedness of quotient, cp. Prop. 2.2.19 of [10])
Let $\mathfrak{I}$ be a closed, two-sided ideal of a C^*-algebra $\mathfrak{A}$. Then:

1. *$\mathfrak{I}$ is self-adjoint, i.e. $\mathfrak{I}^* = \mathfrak{I}$.*

2. *$\mathfrak{U}/\mathfrak{I}$ as defined in the last definition is a C^*-algebra.*

3. *If $\mathfrak{U}$ is unital with unit $\mathbb{1}_{\mathfrak{U}}$, then $\mathfrak{U}/\mathfrak{I}$ is unital with unit $[\mathbb{1}_{\mathfrak{U}}]$.*

Proof. 1.) and 2.) are proven in [10].
3.) Follows directly, since $[\mathbb{1}_{\mathfrak{U}}] \cdot [a] = [\mathbb{1}_{\mathfrak{U}} a] = [a]$ and $[a] \cdot [\mathbb{1}_{\mathfrak{U}}] = [a \mathbb{1}_{\mathfrak{U}}] = [a]$. □

We further have the following very elementary fact:

Remark 1 (Facts regarding the unit)
Let $\mathfrak{U}$ be an unital C^*-algebra with unit $\mathbb{1}_{\mathfrak{U}}$. Then:

1. *$\mathbb{1}^* = \mathbb{1}$*

2. *$\mathbb{1}_{\mathfrak{U}}$ is the unique unit element of $\mathfrak{U}$.*

Proof. 1.) $\mathbb{1}^* A = (A^* \mathbb{1})^* = A^{**} = A$.
2.) Let $\mathbb{1}' \in \mathfrak{U}$ be another unit. Then: $\mathbb{1}' = \mathbb{1}' \mathbb{1}_{\mathfrak{U}} = \mathbb{1}_{\mathfrak{U}}$. □

We now show, that $*$-isomorphisms are automatically unital. But please have in mind, that this does not hold for $*$-morphisms in general!

Remark 2 ($*$-isomorphisms are unital)
Let $\mathfrak{U}_1, \mathfrak{U}_2$ be two C^*-algebras. Let $\phi : \mathfrak{U}_1 \to \mathfrak{U}_2$ be a $*$-isomorphism. Then: ϕ is unital.

Proof. Therefore let $b \in \mathfrak{U}_2$ and $\mathbb{1}_{\mathfrak{U}_1}$ be the unique unit of $\mathfrak{U}_1$. Observe:

$$\begin{aligned}\phi\left(\mathbb{1}_{\mathfrak{U}_1}\right) b &= \phi\left(\mathbb{1}\phi^{-1}(b)\right) = b \\ b\phi\left(\mathbb{1}_{\mathfrak{U}_1}\right) &= \phi\left(\phi^{-1}(b)\mathbb{1}_{\mathfrak{U}_1}\right) = b\end{aligned}$$

And hence $\phi\left(\mathbb{1}_{\mathfrak{U}_1}\right)$ is the unique unit element in $\mathfrak{U}_2$. Hence ϕ is unital. □

We further want to introduce two important examples of C^*-algebras. We will see later, that basically all C^*-algebras arise in the following way:

Example 1 (C^*-algebra of bounded operators on a Hilbert space, cp. Ex. 2.1.2 of [10])
Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$. Define further the operator norm for linear operators on $\mathcal{H}$ via:

$$\|\cdot\|_{\mathcal{B}(\mathcal{H})} := \sup_{f \in \mathcal{H}\setminus\{0\}} \frac{\|\cdot f\|_{\mathcal{H}}}{\|f\|_{\mathcal{H}}}$$

Now definew

$$\mathcal{B}(\mathcal{H}) := \{T : \mathcal{H} \to \mathcal{H} | T \text{ is linear and } \|T\|_{\mathcal{B}(\mathcal{H})} < \infty\}$$

which is equipped with the linear structure

$$\forall \alpha, \beta \in \mathbb{C} : \forall T_1, T_2 \in \mathcal{B}(\mathcal{H}) : \forall f \in \mathcal{H} : (\alpha T_1 + \beta T_2) f := \alpha T_1 f + \beta T_2 f$$

and the algebra structure

$$\forall T_1, T_2 \in \mathcal{B}(\mathcal{H}) : (T_1 T_2) f := (T_1 f)(T_2 f)$$

and an involution given by the adjoint operation. Then $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

Example 2 (C^*-algebra of continuous functions on a compact Hausdorff space, cp. Ex. 2.1.4 of [10])
Let X be a compact Hausdorff space and let $C(X) := C(X, \mathbb{C})$ be the set of complex-valued continuous functions on X. We equip $C(X)$ with a linear structure via:

$$\forall \alpha, \beta \in \mathbb{C} : \forall f_1, f_2 \in C(X) : \forall x \in X : (\alpha f_1 + \beta f_2)(x) := \alpha f_1(x) + \beta f_2(x)$$

Further we equip $C(X)$ with a $*$-algebra structure via:

$$\forall f_1, f_2 \in C(X) : \forall x \in X : (f_1 f_2)(x) := f_1(x) f_2(x) \quad \forall f \in C(X) : f^*(x) := \overline{f(x)}$$

Finally we define the norm on $C(X)$ as the sup norm $\| \cdot \|_\infty$:

$$\forall f \in C(X) : \|f\|_\infty := \sup_{x \in X} |f(x)|$$

Then $C(X)$ is a abelian, unital C^*-algebra with unit element

$$\mathbb{1}_{C(X)} : X \to \mathbb{C}, x \mapsto 1$$

2.2. The Spectrum

We define further the following more advanced notion:

Definition 3 (Spectrum, cp. Ch. 27 of [31])
Let $\mathfrak{U}$ be a C^*-algebra. Then define the spectrum of $\mathfrak{U}$, denoted by $\Delta(\mathfrak{U})$, as:

$$\Delta(\mathfrak{U}) := \{\chi : \mathfrak{U} \to \mathbb{C} | \chi \textit{ is non-zero *-morphism}\}$$

Further elements of $\Delta(\mathfrak{U})$ are called characters.

Further we have the following elementary fact:

Remark 3
Let $\mathfrak{U}$ be an unital C^*-algebra and let $\chi \in \Delta(\mathfrak{U})$. Then: $\chi(\mathbb{1}_\mathfrak{U}) = 1$.

Proof. Since χ is nonzero and linear, it is necessarily surjective. Hence for all $z \in \mathbb{C}$ there is a $a \in \mathfrak{U}$ with $\chi(a) = z$. Hence $z = \chi(a) = \chi(a\mathbb{1}) = \chi(a)\chi(\mathbb{1}_\mathfrak{U}) = z\mathbb{1}$ and with the same argumentation $z = \chi(\mathbb{1}_\mathfrak{U})z$. Hence $\chi(\mathbb{1}_\mathfrak{U}) = 1$. □

2.3. The Case of Abelian Unital C^*-Algebras

We now want to investigate the case of unital, abelian C^*-algebras. One the one hand the theory of those algebras is very beautiful, on the other hand it is one of the cornerstones of this thesis. As the other sections in this crash course, this section is also very brief. A more thorough introduction can be found in [31] and deeper threatments can be found in [19] and [10].

First we want to upgrade the spectrum of an abelian, unital C^*-algebra to a normed space:

Definition 4 (cp. Ch. 27 of [31])
Let $\mathfrak{U}$ be an abelian, unital C^*-algebra with spectrum $\Delta(\mathfrak{U})$. Then define a norm on $\mathfrak{U}$ via:

$$\forall \chi \in \Delta(\mathfrak{U}) : \|\chi\| = \sup_{a \in \mathfrak{U} \setminus \{0\}} \frac{|\chi(a)|}{\|a\|}$$

With this we have:

Proposition 1 (cp. Ch. 27 of [31])
Let $\mathfrak{U}$ be an abelian, unital C^*-algebra with spectrum $\Delta(\mathfrak{U})$. Then:

1. *$\|\cdot\|$ defines a norm on $\mathfrak{U}$.*
2. *$\Delta(\mathfrak{U}) \subset \mathfrak{U}'$, where $\mathfrak{U}'$ is the topological dual of $\mathfrak{U}$.*
3. *$\forall \chi \in \Delta(\mathfrak{U}) : \|\chi\| \leq 1$.*

We use this to topologize $\Delta(\mathfrak{U})$:

Definition 5 (Gel'fand topology, cp. Ch. 27 of [31])
Let $\mathfrak{U}$ be an abelian, unital C^*-algebra with spectrum $\Delta(\mathfrak{U})$. We then define the Gel'fand topology on $\Delta(\mathfrak{U}) \subset \mathfrak{U}'$ as the weak *-topology inherited from $\mathfrak{U}'$.

With this we have the following very important theorem, which establishes the duality between unital, abelian C^*-Algebras and compact Hausdorff spaces:

Theorem 2 (Spectra of abelian, unital C^* algebras are compact and Hausdorff, cp. Ch. 27 of [31])
Let $\mathfrak{U}$ be an abelian, unital C^*-algebra. Then its spectrum $\Delta(\mathfrak{U})$ together with the Gel'fand topology is a compact Hausdorff space.

We now introduce the Gel'fand transform, which makes the duality between abelian, unital C^*-algebras and the algebra of continuous functions on their spectrum explicit:

Definition 6 (Gel'fand transform)
Let $\mathfrak{U}$ be an abelian, unital C^*-algebra. Then the Gel'fand transform is defined as:

$$\mathcal{G} : \mathfrak{U} \to C(\Delta(\mathfrak{U})), a \mapsto (\mathcal{G}(a) : \chi \mapsto \mathcal{G}(a)(\chi) := \chi(a))$$

For convenience the Gel'fand transform is also often denoted by $\check{\cdot}$, i.e.

$$\forall a \in \mathfrak{U} : \forall \chi \in \Delta(\mathfrak{U}) : \check{a}(\chi) := \mathcal{G}(a)(\chi) = \chi(a)$$

We now have the following very important theorem:

Theorem 3 (Gel'fand transform is isomorphism, cp. Ch. 27 [31])
Let $\mathfrak{U}$ be an abelian, unital C^*-algebra. Then the Gel'fand transform is an isometric $*$-isomorphism between $\mathfrak{U}$ and the C^*-algebra of continuous functions $C(\Delta(\mathfrak{U}))$. That it is isometric means explicitly:

$$\forall a \in \mathfrak{U} : \|a\| = \sup_{\chi \in \Delta(\mathfrak{U})} |\chi(a)| = \|\check{a}\|_\infty$$

Further it is automatically unital as a $*$-isomorphism.

Further the following holds for the prototypical example of continuous functions over compact Hausdorff spaces:

Lemma 3 (Spectrum of the algebra of continuous functions, cp. Cor. 27.2.6 of [31])
Let X be a compact Hausdorff space. Let $C(X)$ be the C^*-algebra of continuous functions over X introduced in example 2.
Then: $\Delta\left(C(X)\right) = X$.

2.4. States and Representations

We first introduce the notion of states on C^*-algebras:

Definition 7 (State, cp. [10], [31])
Let $\mathfrak{U}$ be a C^*-algebra. Then a state is a linear functional $\omega \in \mathfrak{U}'$ such that:

1. $\forall a \in \mathfrak{U} : \omega(a^*a) \geq 0$.

2. $\|\omega\| = 1$.

If $\mathfrak{U}$ is unital, we further require, that $\omega(\mathbb{1}_{\mathfrak{U}}) = 1$.

We now introduce the notion of representations of C^*-algebras:

Definition 8 (Representation, cp. [10])
Let $\mathfrak{U}$ be a C^*-algebra. Then a representation of $\mathfrak{U}$ is a tuple $(\mathcal{H}, \pi)$ (or for short π), where $\mathcal{H}$ is a Hilbert space and $\pi : \mathcal{H} \to \mathcal{B}(\mathcal{H})$ is a $*$-morphism.

We now introduce some further properties of representations:

Definition 9 (Further properties of representations, cp. [31], [10])
Let $\mathfrak{U}$ be a C^*-algebra and $(\mathcal{H}, \pi)$ be a representation of $\mathfrak{U}$. Then:

1. *π is called faithful, iff π is injective, i.e. iff $\ker \pi = \{0\}$.*

2. *A linear subspace $\mathcal{H}_1 \subset \mathcal{H}$ is called invariant under π iff $\pi(\mathfrak{U})\mathcal{H}_1 \subseteq \mathcal{H}_1$.*

3. *π is called irreducible, iff there is no non-trivial closed subspace of $\mathcal{H}$ which is invariant under $\pi(\mathfrak{U})$.*

4. *A vector $\Omega \in \mathcal{H}$ is called cyclic for π, iff the set $\{\pi(a)\Omega \subset \mathcal{H} | a \in \mathfrak{U}\}$ is dense in $\mathcal{H}$. In this case we call the tuple $(\mathcal{H}, \pi, \Omega)$ a cyclic representation.*

Further we have as a special case of lemma 1, that representations are norm-decreasing:

Corollar 1
Let $\mathfrak{U}$ be a C^*-algebra and $(\mathcal{H}, \pi)$ be a representation. Then:

$$\forall a \in \mathfrak{U} : \|\pi(a)\| \leq \|a\|$$

Proof. Follows directly with lemma 1. □

We finally want to introduce some further concepts:

Definition 10 (Subrepresentation, trivial part, non-degenerate representation, cp. [10])
Let $\mathfrak{U}$ be a C^*-algebra and $(\mathcal{H}, \pi)$ be a representation. Then:

1. *Let $\mathcal{H}_1 \subseteq \mathcal{H}$ be a closed invariant subspace and P_1 be the projector on $\mathcal{H}_1$. Then the tuple $(\mathcal{H}_1, \pi_1)$ with $\forall a \in \mathfrak{U} : \pi_1(a) = P_1\pi_1(a)P_1$ is a representation of $\mathfrak{U}$, called a subrepresentation.*

2. *The trivial part of $(\mathcal{H}, \pi)$ is the subrepresentation $(\mathcal{H}_0, \pi_0)$ with:*

$$\begin{aligned} \mathcal{H}_0 &:= \{f \in \mathcal{H} | \forall a \in \mathfrak{U} : \pi(a)f = 0\} \\ \pi_0 &= 0 \end{aligned}$$

3. *$(\mathcal{H}, \pi)$ is called non-degenerate iff it has no trivial part, i.e. iff $\mathcal{H}_0 = \{0\}$*

We now want to introduce direct sums of representations:

Definition 11 (Direct sums of representations, cp. [10])
Let $\mathfrak{U}$ be a C^*-algebra and $(\mathcal{H}_i, \pi_i)_{i\in I}$ be a family of representations. We then define the direct sum representation $\left(\bigoplus_{i\in I} \mathcal{H}_i, \bigoplus_{i\in I} \pi_i\right)$ as follows:

1. *$\bigoplus_{i\in I} \mathcal{H}_i$ is the usual direct sum of Hilbert spaces.*

2. *$\pi = \bigoplus_{i\in I} \pi_i$ is such that for all $a \in A$ it holds, that for $f \in \mathcal{H}_i$: $\pi(a)f = \pi_i(a)f$*

Finally we have the following important theorem:

Theorem 4 (cp. Prop. 2.3.6 of [10])
Let $\mathfrak{U}$ be a C^*-algebra and $(\mathcal{H}, \pi)$ be a non-degenerate representation of $\mathfrak{U}$. Then: $(\mathcal{H}, \pi)$ is a direct sum of a family of cyclic representations.

2.5. The GNS Construction

In this section we introduce the Gel'fand-Newmark-Siegel construction, which associates to each unital C^*-algebra and to each state a cyclic representation.

Theorem 5 (Gel'fand-Newmark-Siegel, cp. Ch. 2.3.3 of [10])
Let $\mathfrak{U}$ be an unital C^*-algebra and ω be a state on $\mathfrak{U}$. Then: There exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathfrak{U}$ such that

$$\forall a \in \mathfrak{U} : \omega(a) = \langle \Omega_\omega, \pi_\omega(a)\Omega_\omega \rangle$$

In addition, the representation is unique up to unitary equivalence.

We want to sketch the proof, since it is of great importance.

Proof. We first want to construct a linear space $\mathcal{V}$, an inner product on $\mathcal{V}$ and a vector $\Omega \in \mathcal{V}$, such that above identity holds.

We therefore observe, that $\ker(\omega)$ is an two-sided ideal in $\mathfrak{U}$ such that we set $\mathcal{V} = \mathfrak{U}/\ker(\omega)$. The linear structure on this space is given by $[a] + [b] = [a + b]$ and $\lambda[a] = [\lambda a]$. Then the sesquilinear form defined by

$$\forall a, b \in \mathfrak{U} : \langle [a], [b] \rangle = \omega(a^*b)$$

is well defined and indeed an inner product.

We now guess the vector $\tilde{\Omega}_\omega$ as $\tilde{\Omega}_\omega = [1]$. Further we define define a conjectured representation on $\mathcal{V}$ as:

$$\pi : \mathfrak{U} \to Hom(\mathcal{V}, \mathcal{V}), a \mapsto ([b] \mapsto [ab])$$

We then see directly:

$$\forall a \in \mathfrak{U} : \langle \tilde{\Omega}_\omega, \pi(a)\tilde{\Omega}_\omega \rangle = \omega(a)$$

We then define $\mathcal{H}$ as the completion of $\mathcal{V}$. The corresponding equivalence class of $\tilde{\Omega}_\omega$ will be denoted by Ω_ω. Further the represenation π defined above induces a unique representation on $\mathcal{H}$, which will be denoted by π_ω, since $\mathcal{V} \subset \mathcal{H}$ is dense.

We now want to sketch, how to show uniqueness. Therefore let $(\mathcal{H}'_\omega, \pi'_\omega, \Omega'_\omega)$ be another cyclic representation with above property. We then define a unitary map $U : \mathcal{H}_\omega \to \mathcal{H}'_\omega$ by $\pi_\omega(a)\Omega_\omega \mapsto \pi'_\omega(a)\Omega$. It can be indeed shown, that this defines a unitary transformation. □

This gives directly the following easy corollar:

Corollar 2 (Uniqueness of the GNS representation)
Let $(\mathcal{H}, \pi, \Omega)$ be a cyclic representation. Then: $(\mathcal{H}, \pi, \Omega)$ is unitary equivalent to the GNS-representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ with ω defined via:

$$\forall a \in \mathfrak{U} : \omega(a) = \langle \Omega, \pi(a)\Omega \rangle$$

Proof. For this to hold, ω must define a state. That this is indeed the case is shown in Ch. 2.3.2 of [10]. □

Finally we have the following structure theorem:

Theorem 6 (Structure theorem for representations of C^*-algebras)
Let $\mathfrak{U}$ be a C^*-algebra and $(\mathcal{H}, \pi)$ be a non-degenerate representation. Then: There exists a family of states $(\omega_i)_{i \in I}$ on $\mathfrak{U}$ such that

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_{\omega_i}$$
$$\pi = \bigoplus_{i \in I} \pi_{\omega_i}$$

where $(\mathcal{H}_{\omega_i}, \pi_{\omega_i}, \Omega_{\omega_i})_{i \in I}$ are the corresponding GNS-representations.

Proof. Follows directly with corollar 2 and theorem 4. □

2.6. Spectral Theory of C^*-Algebras

In this section we want to prove a spectral theorem for C^*-algebras. Therefore first recall the Riesz-Markov theorem:

Theorem 7 (Riesz-Markov, cp. Cor. 25.1.16 of [31])
Let X be a compact Hausdorff space. Let $\Lambda : C(X) \to \mathbb{C}$ be a positive, continuous, linear functional with $\Lambda(\mathbb{1}_{C(X)}) = 1$.

Then: There is a unique regular, Borel probability measure μ on X together with the Borel σ-algebra on X, such that:

$$\forall f \in C(X) : \Lambda(f) = \int_X f d\mu$$

Now we prove first a Lemma, which incorporates the basic statement of the later proven spectral theorem and is basically a special case of the GNS representation:

Lemma 4 (cp. Ch. 29.2 of [31])
Let $\mathfrak{A}$ be an unital, abelian C^*-algebra and let ω be a state on $\mathfrak{A}$.
Then: The tuple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ with

1. $\mathcal{H}_\omega = L^2(\Delta(\mathfrak{A}), d\mu_\omega)$ *with $d\mu_\omega$ being the spectral measure defined via*

$$\forall b \in \mathfrak{A} : \int_{\Delta(\mathfrak{A})} d\mu_\omega \check{b} = \omega(b)$$

, where $\check{\cdot}$ denotes the Gel'fand-transform,

2. $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_\omega), a \mapsto \pi(a)$ *with* $\pi(a) : f \mapsto \check{b} \cdot f$*,*

3. $\Omega_\omega = \mathbb{1}_{C(\Delta(\mathfrak{A}))}$

is a cyclic representation. Further any cyclic representation $(\mathcal{H}', \pi', \Omega')$ which satisfies

$$\forall a \in \mathfrak{A} : \omega(a) = \langle \Omega', \pi'(a)\Omega' \rangle$$

(especially the GNS-representation) is unitarily equivalent to this representation.

Proof. With Riesz-Markov Theorem, we can define the space $L^2(\Delta(\mathfrak{A}), d\mu_\omega)$, where the Measure is defined via

$$\forall f \in C(\Delta(\mathfrak{A})) : \int_{\Delta(\mathfrak{A})} d\mu_\omega f = \omega(\mathcal{G}^{-1} f) \tag{2.1}$$

and where $\mathcal{G}^{-1}$ denotes the inverse Gel'fand transform. It can be easily shown, that $\omega \circ \mathcal{G}^{-1}$ is a positive, continuous, linear functional. Since the Gel'fand transform is an isomorphism, eq. (2.1) is equivalent to:

$$\forall b \in \mathfrak{A} : \int_{\Delta(\mathfrak{A})} d\mu_\omega \check{b} = \omega(b)$$

We now show, that

$$\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}), a \mapsto \pi(a)$$
$$\pi(a) : \mathcal{H} \to \mathcal{H}, f \mapsto \check{b} \cdot f$$

defines a representation. Therefore observe first, that for all $a \in \mathfrak{A}$ it holds, that $\pi(a)$ is bounded:

Let $f \in \mathcal{H} = L^2(\Delta(\mathfrak{A}), d\mu_\Omega)$. Then:

$$\begin{aligned} \|\pi(a)f\|^2 &= \int_{\Delta(\mathfrak{A})} d\mu_\Omega |\check{a} f|^2 \\ &= \int_{\Delta(\mathfrak{A})} d\mu_\Omega |f|^2 |\check{a}|^2 \\ &\leq \|\check{a}\|_\infty^2 \|f\|^2 \end{aligned}$$

And hence $\pi(a) \in \mathcal{B}(\mathcal{H})$. Further observe, that π is a $*$-morphism, since the Gel'fand transform is a $*$-morphism.

We now show, that $\Omega_\omega = \mathbb{1}_{C(\Delta(\mathfrak{U}))} \in L^2(\Delta(\mathfrak{U}), d\mu_\omega)$ is cyclic for $\pi(\mathfrak{U})$. But this holds directly, because of $\pi(\mathfrak{U}) \cdot \mathbb{1}_{C(\Delta(\mathfrak{U}))} = C(\Delta(\mathfrak{U})) \subset L^2(\Delta(\mathfrak{U}), d\mu_\Omega)$, which holds, since the Gel'fand transformation is a $*$-isomorphism.

Unitary equivalence follows directly with corollar 2, since the following holds:

$$\forall a \in \mathfrak{U} : \langle \Omega_\omega, \pi_\omega(a)\Omega_\omega \rangle = \int_{\Delta(\mathfrak{U})} d\mu_\omega \mathbb{1}_{C(\Delta(\mathfrak{U}))} \check{a} \mathbb{1}_{C(\Delta(\mathfrak{U}))} = \omega(a)$$

This completes the proof. □

Now recall from theorem 4, that each representation is unitarily equivalent to a direct sum of cyclic representations. With this and with lemma 4 the following "spectral theorem" follows directly:

Theorem 8 (Spectral theorem for abelian, unital C^*-algebras, cp. Ch. 29.2 of [31])
Let $(\mathfrak{U}, \pi)$ be an abelian, unital C^*-algebra and let $(\mathcal{H}, \pi)$ be a non-degenerate representation. Then: There is a family of states $(\omega_i)_{i \in I}$ such that

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_{\omega_i}$$
$$\pi = \bigoplus_{i \in I} \pi_{\omega_i}$$

where $(\mathcal{H}_{\omega_i}, \pi_{\omega_i}, \Omega_{\omega_i})$ denotes the cyclic representation from lemma 4 corresponding to the state ω_i.

Proof. Theorem 4 gives, that we have a family of cyclic representations $(\mathcal{H}_i, \pi_i, \Omega_i)_{i \in I}$ such that

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_{\omega_i}$$
$$\pi = \bigoplus_{i \in I} \pi_{\omega_i}$$

holds. From corollar 2 we know, that ω_i with $\forall a \in \mathfrak{U} : \omega_i(a) := \langle \Omega_i, \pi_i(\omega_i)\Omega_i \rangle$ defines indeed a state on $\mathfrak{U}$. Further we have with this and with lemma 4, that those cyclic representations are indeed isomorphic to the cyclic representations $(\mathcal{H}_{\omega_i}, \pi_{\omega_i}, \Omega_{\omega_i})$ defined in lemma 4. □

Finally, motivated by the outcome of lemma 4 and theorem 8, we want to give this class of representations a name:

Definition 12 (Gel'fand representation)
Let $\mathfrak{U}$ be an abelian, unital C^*-Algebra with spectrum $\Delta(\mathfrak{U})$. Let $d\mu$ be a regular, Borel probalbility measure on $\Delta(\mathfrak{U})$. Then a representation $(L^2(\Delta(\mathfrak{U}), d\mu), \pi)$ of $\mathfrak{U}$

$$\pi : \mathfrak{U} \to \mathcal{B}(\mathcal{H}_\omega), a \mapsto \pi(a)$$

with

$$\pi(a) : f \mapsto \check{b} \cdot f$$

is called a Gel'fand representation of $\mathfrak{U}$.

The purpose of chapter 3 is now to analyze, to which extent above spectral theorem is compatible with so called inductive limits of C^*-algebras and projective limits of compact Hausdorff spaces.

3. Spectral Theory of Inductive Limit C^*-Algebras

This chapter provides a thorough treatment of the spectral theory of inductive limit C^*-algebras. Therefore in the first section inductive and projective limits in arbitrary categories are introduced and as important examples the cases of Banach spaces and compact Hausdorff spaces are analyzed. In the next section the case of Hilbert spaces, orthonormal bases and operators thereon are analyzed. The third section introduces the concept of inductive limits of C^*-algebras. Further it is shown, how the spectrum of the inductive limit C^*-algebra can be expressed in terms of the spectra of the inductive family. In the next section the case of continuous functions on compact Hausdorff spaces is considered and concepts that appeared in the earlier sections are illustrated at this important example. The next section investigates the inductive limit of L^2 spaces and dually the projective limit of measure spaces. Finally it is shown, that the L^2 space over a projective limit of measure spaces is isometrically isomorphic to the inductive limit of the L^2 spaces over the projective family of measure spaces. The next section finally introduces inductive limits of states and representations and examines the behaviour of the Gel'fand transform under inductive limits. In the last section finally versions of lemma 4 and theorem 8 are proven for inductive limits of C^*-algebras.

For the first two chapters the main references are given by [31], [27]. The basic notions presented in the third and sixth section can be also found in [18] and [35]. The basic concepts of the fifth sections can be found in [31]. Since the appearing concepts are the basis of the proof of our final spectral theorem, all proofs are performed in depth. Further theorems without a reference are new to the best of the author's knowledge. A proof of theorem 18 can be found also in [30], but the proof presented here is slightly different and more detailed. Further important results of this section are the equivalence of different notions of cylindrical functions (lemma 14), the compatibility of L^2 spaces with projective and inductive limits (theorem 22), the compatibility of the Gel'fand transform and Gel'fand representation with inductive limits (lemma 17 and theorem 25) and finally the compatibility of lemma 4 and theorem 8 with inductive and projective limits, which is proven in the last section.

Finally the author wants to emphasize, that in this section the morphisms in the category of compact Hausdorff spaces are considered as surjective and the morphisms in the category of Hilbert spaces and C^*-algebras are dually considered to be isometries. The more general case of arbitrary continuous maps in the first case and contracting maps in the second case needs further investigation.

3.1. Inductive and Projective Limits

In this section the notions of inductive and projective limits are introduced for arbitrary categories. Afterwards we will prove the existence of inductive limits in the category of

Banach spaces and the existence of projective limits in the category of compact Hausdorff spaces. Those proofs will be performed elaborately, since they are the basis of all proofs in this section. In the case of Banach spaces we define the morphisms of this category as isometries, since we consider also only the case of isometries later on and in our application to loop quantum gravity and polymer quantization.

3.1.1. Inductive and Projective Limits in Arbitrary Categories

We first define the following notion:

Definition 13 (Label Set, cp. Ch. 1 of [27])
A label set is a tuple $(L, \leq)$, that is a partially ordered, directed set. This means:

1. *$\leq$ is reflexive, symmetric and transitive.*
2. *$\forall \gamma, \gamma' \in L : \exists \hat{\gamma} \in L : \gamma, \gamma' \leq \hat{\gamma}$.*

We now define the notions of inductive systems and inductive limits:

Definition 14 (Inductive system, inductive limit, cp. [22], [27], [31])
Let $\mathcal{C}$ be a category and $(L, \leq)$ be a label set. Then:

1. *A family of tuples $(X_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ is called an inductive system, iff*
   a) *$\forall \gamma \in L : X_\gamma \in \mathrm{obj}(\mathcal{C})$.*
   b) *$\forall \gamma' \geq \gamma \in L : \phi_{\gamma'\gamma} \in \mathrm{mor}(X_\gamma, X_{\gamma'})$ s.th.:*
      i. *$\forall \gamma \in L : \phi_{\gamma\gamma} = id$.*
      ii. *$\forall \gamma'' \geq \gamma' \geq \gamma \in L : \phi_{\gamma''\gamma'} \circ \phi_{\gamma'\gamma} = \phi_{\gamma''\gamma}$.*
2. *An object $\lim_{\rightarrow} X_\gamma \in \mathrm{obj}(\mathcal{C})$ together with a family of morphisms $(\phi_\gamma \in \mathrm{mor}(X_\gamma, \lim_{\rightarrow} X_\gamma))_{\gamma \in L}$ is called the inductive limit of an inductive system $(X_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ iff*
   a) *$\forall \gamma' \geq \gamma \in L : \phi_{\gamma'} \circ \phi_{\gamma'\gamma} = \phi_\gamma$*
   b) *For each other $Y \in \mathrm{obj}(\mathcal{C})$ together with a family of morphisms $(\tilde{\phi}_\gamma \in \mathrm{mor}(X_\gamma, Y))_{\gamma \in L}$, such that this property holds, there is a unique morphism $u \in \mathrm{mor}(\lim_{\rightarrow} X_\gamma, Y)$, such that anything commutes (universal property).*

The dual concept is the concept of projective families and projective limits:

Definition 15 (Projective system, projective limit, cp. [22], [27], [31])
Let $\mathcal{C}$ be a category and $(L, \leq)$ be a label set. Then:

1. *A tuple $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ is called projective system, iff*
   a) *$\forall \gamma \in L : X_\gamma \in \mathrm{obj}(\mathcal{C})$.*
   b) *$\forall \gamma' \geq \gamma \in L : p_{\gamma\gamma'} \in \mathrm{mor}(X_{\gamma'}, X_\gamma)$ s.th.:*
      i. *$\forall \gamma \in L : p_{\gamma\gamma} = id$.*
      ii. *$\forall \gamma'' \geq \gamma' \geq \gamma \in L : p_{\gamma\gamma'} \circ p_{\gamma'\gamma''} = p_{\gamma\gamma''}$.*
2. *An object $\lim_{\leftarrow} X_\gamma \in obj(\mathcal{C})$ together with a family of morphisms $(p_\gamma \in \mathrm{mor}(\lim_{\leftarrow} X_\gamma, X_\gamma))_{\gamma \in L}$ is called the projective limit of a projective system system $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ iff*
   a) *$\forall \gamma' \geq \gamma \in L : p_\gamma = p_{\gamma\gamma'} \circ p_{\gamma'}$*

b) For each other $Y \in \mathrm{obj}(\mathcal{C})$ together with a family of morphisms $(\tilde{p}_\gamma \in mor(Y, X_\gamma))_{\gamma\in L}$ such that this property holds, there is a unique morphism $u \in \mathrm{mor}(Y, \lim_{\leftarrow} X_\gamma)$, such that anything commutes (universal property).

We now want to show the following theorem:

Theorem 9 (Inductive/Projective Limits are Unique, cp. Ch. III.3 and Ch. III.4 of [22]) *Let $\mathcal{C}$ be a category. Then:*

1. *Let $(X_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive system in $\mathcal{C}$. Then: If its inductive limit exists, then it is unique.*
2. *Let $(X_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be a projective system in $\mathcal{C}$. Then: If its projective limit exists, then it is unique.*

Proof. 1.) We now show, that the inductive limit is unique. Therefore let $(X, (\tilde{\phi}_\gamma)_{\gamma\in L})$, $(Y, \bar{\phi}_\gamma)_{\gamma\in L})$ be two inductive limits of above inductive family. By the universal property we have two unique universal morphisms $u : Y \to X$ and $\tilde{u} : X \to Y$. Further we have by the universal property two unique universal morphisms $a : X \to X$ and $b : Y \to Y$. Since the identity maps $id_X : X \to X$ and $id_Y : Y \to Y$ satisfy the universal property, we have, that $a = id_X$ and $b = id_Y$. Further we have, that the maps $\tilde{u} \circ u : Y \to Y$ and $u \circ \tilde{u} : X \to X$ satisfy the universal property, and hence we have, that $\tilde{u} \circ u = id_Y$ and $u \circ \tilde{u} = id_X$. Hence u is an isomorphism in $\mathcal{C}$ with inverse $\tilde{u}$.

2.) We now show, that the projective limit is unique. Therefore let $(X, (\tilde{p}_\gamma)_{\gamma\in L})$, $(Y, (\bar{p}_\gamma))_{\gamma\in L}$ be two projective limits of above projective family. By the universal property we have two unique universal morphisms $u : X \to Y$ and $\tilde{u} : Y \to X$. Further we have by the universal property two unique universal morphisms $a : X \to X$, $b : Y \to Y$. Since the identity maps $id_X : X \to X$ and $id_Y : Y \to Y$ satisfy the universal property, we have, that $a = id_X$ and $b = id_Y$. Further we have, that the maps $\tilde{u} \circ u : X \to X$ and $u \circ \tilde{u} : Y \to Y$ satisfy the universal property, and hence we have, that $\tilde{u} \circ u = id_X$ and $u \circ \tilde{u} = id_Y$. Hence u is an isomorphism in $\mathcal{C}$ with inverse $\tilde{u}$. □

3.1.2. Inductive Limits in the Category of Banach Spaces

In this paragraph we want to investigate inductive limits in the category of Banach spaces. This is very important for the rest of this thesis, since Hilbert spaces and C^*-algebras are both Banach spaces.

But first we need two Lemmata. The first of those is the well known bounded linear transformation theorem:

Lemma 5 (Bounded Linear Transformation Theorem, cp. Thm. II.1.5 of [36]) *Let B_1, B_2 be Banach spaces and let $V_1 \subset B_1$ be a dense subset. Let further $T : B_1 \to B_2$ be a bounded linear operator. Then there is a unique bounded extension $\tilde{T} : B_1 \to B_2$. Further $\|\tilde{T}\| = \|T\|$ and $\tilde{T}$ is an isometry, if T is an isometry.*

Proof. Let $f \in B_1$ and $(f_n)_{n\in\mathbb{N}} \subset V_1$ be a sequence with $\lim_{n\to\infty} f_n = f$. Then $\tilde{T}$ is defined via:

$$\tilde{T}f := \lim_{n\to\infty} Tf_n$$

That this really defines a bounded extension, that $\|T\| = \|\tilde{T}\|$ holds and that the extension is unique can be shown easily (cp. [36]) . That the extension is an isometry if T is an isometry follows, since the norm is continuous on Banach spaces:

$$\begin{aligned}\|\tilde{T}f\| &= \|\lim_{n\to\infty} Tf_n\| \\ &= \lim_{n\to\infty} \|Tf_n\| \\ &= \lim_{n\to\infty} \|f_n\| \\ &= \|f\|\end{aligned}$$

□

In the following we will often refer to this theorem just by "BLT". For later use we need further the following special case of above theorem, which gives also a criterion for bijectivity of the unique, bounded extension:

Lemma 6
Let B_1, B_2 be Banach spaces and let $V_1 \subset B_1$, $V_2 \subset B_2$ be dense subsets. Let further $T : V_1 \to V_2$ be a bounded, bijective linear isometry. Then the unique bounded extension $\tilde{T} : B_1 \to B_2$ from the BLT theorem is a bijective, linear isometry.

Proof. Please recall from the proof of the last theorem, that the unique bounded extension $\tilde{T}$ of T is given by the following construction: Let $f \in B_1$ and $(f_n)_{n\in\mathbb{N}} \subset V_1$ with $\lim_{n\to\infty} f_n = f$. Then $\tilde{T}f := \lim_{n\to\infty} Tf_n$.
First observe, that the unique extension $\tilde{T}$ is an isometry by lemma 5.
Recall now, that isometries are automatically injective (cp. Problem 2.5 of [21]). Since the unique extension $\tilde{T}$ is an isometry, we then have directly, that $\tilde{T}$ is injective.
We now show surjectivity: Let $g \in B_2$. Then there is a sequence $(g_n)_{n\in\mathbb{N}} \subset V_2$ with $g_n \to g$, since $V_2 \subset B_2$ is dense. Then $T^{-1}g_n = f_n$ defines a sequence in V_1 by bijectivity of T. Since T is an isometry, we further have, that f_n is Cauchy in B_1, and hence converges in B_1. We now have:

$$\tilde{T}\left(\lim_{n\to\infty} f_n\right) := \lim_{n\to\infty} Tf_n = \lim_{n\to\infty} TT^{-1}g_n = \lim_{n\to\infty} g_n = g$$

and hence surjectivity of $\tilde{T}$ follows. Hence the claim follows. □

Now we investigate the inductive limit in the category of Banach spaces.

Theorem 10 (cp. Ex. 11.5.26 of [18] for a Hilbert space version)
Let Ban be the category whose objects are Banach spaces and whose morphisms are isometries. Then the inductive limit exists in this category and is unique.

Proof. Let $(B_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive family of Banach spaces.

Assertion 1: Set $\mathfrak{X} := \{(f_\gamma)_{\gamma\in L} \in \prod_{\gamma\in L} B_\gamma | \sup_{\gamma\in L} \|f_\gamma\|_\gamma < \infty\}$ and define a norm on $\mathfrak{X}$ via $\|\cdot\| : \mathfrak{X} \to \mathbb{R}, (f_\gamma)_{\gamma\in L} \mapsto \sup_{\gamma\in L} \|f_\gamma\|_\gamma$ and a linear structure by $(\alpha f_\gamma + \beta g_\gamma)_{\gamma\in L} = \alpha(f_\gamma)_{\gamma\in L} + \beta(g_\gamma)_{\gamma\in L}$. Then: $\mathfrak{X}$ is a Banach space.

Proof of Assertion 1:
That $\|\cdot\|$ defines a norm on $\mathfrak{X}$ can be shown easily.
We first show the following statement: Let $\left((f_\gamma^m)_{\gamma\in L}\right)_{m\in\mathbb{N}}$ be a Cauchy sequence in $\mathfrak{X}$.

Then f_γ^m is a Cauchy sequence in B_γ for all $\gamma \in L$.
Therefore let $\big((f_\gamma^m)_{\gamma\in L}\big)_{m\in\mathbb{N}} \subset \mathfrak{X}$ be Cauchy, i.e.:

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall m, n > N : \left\| \big((f_\gamma^m)_{\gamma\in L}\big)_m - \big((f_\gamma^m)_{\gamma\in L}\big)_n \right\| \leq \epsilon$$

Then the following holds:

$$\|f_\gamma^m - f_\gamma^n\|_\gamma \leq \sup_{\gamma\in L} \|f_\gamma^m - f_\gamma^n\|_\gamma = \left\| \big((f_\gamma^m)_{\gamma\in L}\big)_m - \big((f_\gamma^m)_{\gamma\in L}\big)_n \right\| \leq \epsilon$$

Hence $(f_\gamma^m)_{m\in\mathbb{N}}$ is a Cauchy sequence in B_γ for all $\gamma \in L$.
We now show, that a Cauchy sequence in $\mathfrak{X}$ whose components in B_γ converge towards a $g_\gamma \in B_\gamma$ converges towards $(g_\gamma)_{\gamma\in L}$ in $\mathfrak{X}$. The argument is similar to the argument, that the space of bounded functions with values in a Banach space is complete in the sup norm: Let $\epsilon > 0$. Then there is a $N \in \mathbb{N}$ such that

$$\left\| \big((f_\gamma^m)_{\gamma\in L}\big)_m - \big((f_\gamma^m)_{\gamma\in L}\big)_n \right\| < \epsilon$$

for all $m, n > N$. Further there is for any $\gamma \in L$ a $m_\gamma > N$ such that

$$\left\| f_\gamma^{m_\gamma} - g_\gamma \right\|_\gamma < \epsilon$$

Hence we have for all $\gamma \in L$ and for each $n > N$:

$$\begin{aligned}
\|f_\gamma^n - g_\gamma\|_\gamma &\leq \|f_\gamma^m - f_\gamma^{m_\gamma}\|_\gamma + \|f_\gamma^{m_\gamma} - g_\gamma\|_\gamma \\
&\leq \left\| \big((f_\gamma^m)_{\gamma\in L}\big)_m - \big((f_\gamma^m)_{\gamma\in L}\big)_{m_\gamma} \right\| + \|f_\gamma^{m_\gamma} - g_\gamma\|_\gamma \\
&\leq 2\epsilon
\end{aligned}$$

and hence $\mathfrak{X}$ is complete and hence a Banach space. Hence Assertion 1 holds.

Assertion 2: Define a map $j_\gamma : B_\gamma \to \mathfrak{X}$ for all $\gamma \in L$ via

$$j_\gamma : B_\gamma \to \mathfrak{X}, f_\gamma \mapsto (f_\beta)_{\beta\in L}$$

where

$$f_\beta = \begin{cases} \psi_{\beta\gamma} f_\gamma & \beta \geq \gamma \\ 0 & \text{else} \end{cases}$$

Then: j_γ is an isometric, linear embedding.

Proof of Assertion 2: We first show, that j_γ is an isometry: Since $\psi_{\gamma'\gamma}$ is an isometry we have

$$\forall \beta \geq \gamma : \|f_\beta\|_\beta = \|\psi_{\beta\gamma} f_\gamma\|_\beta = \|f_\gamma\|_\gamma \geq 0$$

further for all other β we have, that $\|f_\beta\|_\beta = 0$ holds. Hence we have

$$\|j_\gamma f_\gamma\| = \|(f_\beta)_{\beta\in L}\| = \sup_{\beta\in L} \|f_\beta\|_\beta = \|f_\gamma\|_\gamma$$

Hence j_γ is an isometry. By this we also have, that it j_γ is injective. Linearity follows also directly, since $\psi_{\gamma'\gamma}$ is linear. Hence Assertion 2 holds.

Please observe, that j_γ does not satisfy $j_{\gamma'} \circ \psi_{\gamma'\gamma} = j_\gamma$. Therefore we want to find a closed subspace $\mathfrak{X}_0$, s.th. j_γ satisfies this condition on the quotient space!

Assertion 3: Set $\mathfrak{X}_0 = \{(f_\gamma)_{\gamma\in L} \in \mathfrak{X} | \text{The net } (\|f_\gamma\|_\gamma)_{\gamma\in L} \subset [0,\infty) \text{ converges to } 0\}$. Then $\mathfrak{X}_0 \subset \mathfrak{X}$ is a closed subspace.

If this assertion is true, we have, that $\mathfrak{X}/\mathfrak{X}_0$ is a Banach space with respect to the usual norm on quotient spaces $\|\cdot\|_{\mathfrak{X}/\mathfrak{X}_0} = \inf_{y\in\mathfrak{X}_0} \|\cdot - y\|_{\mathfrak{X}}$ (cp. Ch. I.3 of [36]).

Proof of Assertion 3: We first show, that it is a linear subspace: Let $(f_\gamma), (g_\gamma) \in \mathfrak{X}_0$ and $\alpha, \beta \in \mathbb{C}$. Then:

$$0 \le \|\alpha f_\gamma + \beta g_\gamma\| \le |\alpha|\|f_\gamma\|_\gamma + |\beta|\|g_\gamma\|_\gamma \to 0$$

Hence it is a subspace. We now show, that it is closed. Therefore let $(f_\gamma)_{\gamma\in L} \in \mathfrak{X}\setminus\mathfrak{X}_0$, i.e.

$$\begin{aligned}
&\sup_{\gamma\in L} \|f_\gamma\|_{\gamma\in L} < \infty \wedge \\
&\exists \tilde{\epsilon} > 0 : \forall \gamma_0 \in L : \exists \gamma \ge \gamma_0 : \|f_\gamma\|_\gamma \ge \tilde{\epsilon}
\end{aligned} \tag{3.1}$$

Now define the following neighborhood of $(f_\gamma)_{\gamma\in L}$:

$$(f_\gamma)_{\gamma\in L} \in U := \left(\prod_{\gamma\in L} U_{\frac{\tilde{\epsilon}}{2}}(f_\gamma)\right) \cap \mathfrak{X}$$

Please observe, that U is open in $\mathfrak{X}$, since it is the restriction of a product of open sets. We now want to show, that $U \cap \mathfrak{X}_0 = \emptyset$. Therefore let $(g_\gamma)_{\gamma\in L} \in U$. We then have $\forall \gamma \in L : g_\gamma \in U_{\frac{\tilde{\epsilon}}{2}}(f_\gamma)$ and with this:

$$\begin{aligned}
\forall \gamma \in L : \|g_\gamma\|_\gamma &= \|g_\gamma - f_\gamma + f_\gamma\| \\
&\ge |\|f_\gamma\| - \|g_\gamma - f_\gamma\|\|
\end{aligned} \tag{3.2}$$

Now let $\gamma_0 \in L$. We then have $\gamma \ge \gamma_0$ s.th. $\|f_\gamma\|_\gamma \ge \tilde{\epsilon}$ by eq. (3.1). Further we have $\|g_\gamma - f_\gamma\|_\gamma < \frac{\tilde{\epsilon}}{2}$, since $g_\gamma \in U_{\frac{\tilde{\epsilon}}{2}}(f_\gamma)$. With this and with eq. (3.2) we obtain:

$$\|g_\gamma\|_\gamma \ge \tilde{\epsilon} - \frac{\tilde{\epsilon}}{2} = \frac{\tilde{\epsilon}}{2}$$

And hence we have shown:

$$\exists \epsilon > 0 : \forall \gamma_0 \in L : \exists \gamma \ge \gamma_0 : \|g_\gamma\|_\gamma \ge \epsilon$$

Hence $(g_\gamma)_{\gamma\in L} \notin \mathfrak{X}_0$, and hence $\mathfrak{X}_0$ is closed.

Assertion 4: Let $\pi : \mathfrak{X} \to \mathfrak{X}/\mathfrak{X}_0$ be the canonical projection. Define $\psi_\gamma : B_\gamma \to \mathfrak{X}/\mathfrak{X}_0$ via $\psi_\gamma = \pi \circ j_\gamma$. Then ψ_γ is a linear isometry and $\psi_{\gamma'} \circ \psi_{\gamma'\gamma} = \psi_\gamma$.

Proof of Assertion 4: We first show, that for $f_\gamma \in X_\gamma$ it holds, that $j_\gamma f_\gamma - j_{\gamma'}\psi_{\gamma'\gamma} f_\gamma \in \mathfrak{X}_0$, since then $\psi'_\gamma \circ \psi_{\gamma'\gamma} = \psi_\gamma$ follows.

Let $f_\gamma \in B_\gamma$. Then $j_\gamma f_\gamma = (f_\beta)_{\beta\in L}$ with

$$f_\beta = \begin{cases} \psi_{\beta\gamma} f_\gamma & \beta \ge \gamma \\ 0 & \text{else} \end{cases}$$

Further we have $j_{\gamma'} \circ \psi_{\gamma'\gamma} f_\gamma = (\tilde{f}_\beta)_{\beta \in L}$ with

$$\tilde{f}_\beta = \begin{cases} \psi_{\beta\gamma'}\psi_{\gamma'\gamma} f_\gamma & \forall \beta \geq \gamma' \\ 0 & \forall \gamma' > \beta \geq \gamma \\ 0 & \text{else} \end{cases}$$

Hence we have

$$(f_\beta - \tilde{f}_\beta) = \begin{cases} 0 & \forall \beta > \gamma' \\ \psi_{\beta\gamma} f_\gamma & \forall \gamma' > \beta \geq \gamma \\ 0 & \text{else} \end{cases}$$

Hence:

$$\|f_\beta - \tilde{f}_\beta\|_\beta = \begin{cases} 0 & \forall \beta > \gamma' \\ \|f_\gamma\|_\gamma & \forall \gamma' > \beta \geq \gamma \\ 0 & \text{else} \end{cases}$$

And hence $f_\beta - \tilde{f}_\beta \in \mathfrak{X}_0$.
Now we want to show, that ψ_γ is an isometry. Therefore we first show: $\forall \epsilon > 0 : \forall \gamma \in L : \forall f_\gamma \in B_\gamma : \exists (g_\gamma)_{\gamma \in L} \in \mathfrak{X}_0 : \|j_\gamma f_\gamma - (g_\gamma)_{\gamma \in L}\| > \|f_\gamma\|_\gamma - \epsilon$. Therefore let $(\tilde{g}_\gamma) \in \mathfrak{X}_0$. Hence we have that $\forall \epsilon > 0 : \exists \gamma_0 \in L : \forall \gamma \geq \gamma_0 : \|\tilde{g}_\gamma\|_\gamma < \epsilon$. Now define $(g_\gamma)_{\gamma \in L} \in \mathfrak{X}_0$ as:

$$g_\gamma = \begin{cases} \tilde{g}_\gamma & \gamma \geq \gamma_0 \\ 0 & \text{else} \end{cases}$$

Then we have $\|(g_\gamma)_{\gamma \in L}\|_{\mathfrak{X}} < \epsilon$ and hence:

$$\begin{aligned} \|j_\gamma f_\gamma - (g_\gamma)_{\gamma \in L}\|_{\mathfrak{X}} &\geq |\|j_\gamma f_\gamma\|_{\mathfrak{X}} - \|(g_\gamma)_{\gamma \in L}\|_{\mathfrak{X}}| \\ &\geq |\|j_\gamma f_\gamma\|_{\mathfrak{X}} - \epsilon| \\ &= |\|f_\gamma\|_\gamma - \epsilon| \end{aligned}$$

On the other hand we obtain with the same argumentation $\|j_\gamma f_\gamma - (g_\gamma)_{\gamma \in L}\|_{\mathfrak{X}} \leq \|f\gamma\|_\gamma + \epsilon$ and hence we have for all $\epsilon > 0$:

$$\|f_\gamma\|_\gamma + \epsilon > \inf_{g \in \mathfrak{X}_0} \|j_\gamma f_\gamma - (g_\gamma)_{\gamma \in L}\| > \|f_\gamma\|_\gamma - \epsilon$$

and hence ψ_γ is an isometry. Linearity follows directly with the linearity of j_γ and the linearity of π and hence Assertion 4 holds.

Assertion 5: $\mathrm{im}(\psi_\gamma) \subset \mathfrak{X}/\mathfrak{X}_0$ is a closed subspace and $\forall \gamma' \geq \gamma : \mathrm{im}(\psi_\gamma) \subseteq \mathrm{im}(\psi_{\gamma'})$.

Proof of Assertion 5: Since $\psi_{\gamma'}$ is an isometry and B_γ is complete, we have, that $\mathrm{im}(\psi_\gamma)$ is complete and hence a closed subspace. Let further $\gamma' \geq \gamma$. Then $\psi_{\gamma'} \circ \psi_{\gamma'\gamma} = \psi_\gamma$. Let $f_\gamma \in B_\gamma$. Then:

$$\psi_\gamma(f_\gamma) = \psi_{\gamma'}(\psi_{\gamma'\gamma} f_\gamma)$$

And hence we have $\psi_\gamma(f_\gamma) \in \mathrm{im}(\psi_{\gamma'})$ and hence $\mathrm{im}(\psi_\gamma) \subset \mathrm{im}(\psi_{\gamma'})$.

Assertion 6 (basically a Lemma): Let L be a Label set and D be a Banach space. Let $(D_\gamma)_{\gamma \in L} \subset D$ be a family of closed subspsaces with $\forall \gamma' \geq \gamma : D_\gamma \subseteq D_{\gamma'}$. Then:

$$\overline{\bigcup_{\gamma \in L} D_\gamma} \subset D$$

is a closed subspace and hence defines a Banach space.

Proof of Assertion 6:
Since $\forall \gamma \in L : D_\gamma \subset D$ we have $\bigcup_{\gamma\in L} D_\gamma \subset D$. Since D is closed, we have, that $\overline{\bigcup_{\gamma\in L} D_\gamma} \subset D$ is a closed subspace of a Banach space. Hence it is a Banach space.

Assertion 7: Let A be a Banach space together with maps $\tilde{\psi}_\gamma : B_\gamma \to A$ with $\forall \gamma' \geq \gamma : \tilde{\psi}_{\gamma'} \circ \psi_{\gamma'\gamma} = \tilde{\psi}_\gamma$. Define further $\lim_\to B_\gamma := \overline{\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)}$. Then there is a unique morphism $u : \lim_\to B_\gamma \to A$ such that everything commutes.

Proof of Assertion 7: We define the conjectured universal morphism on the dense set $\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma) \subset \lim_\to B_\gamma$. Let $f \in \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$. Hence there is a $\gamma \in L$ with $f \in \mathrm{im}(\psi_\gamma)$. Since ψ_γ is isometry, it is injective and hence there is a unique $f_\gamma \in B_\gamma$ with $\psi_\gamma(f_\gamma) = f$. Then set $u(f) = \tilde{\psi}_\gamma(f_\gamma)$.

We now show, that this is independent of γ: Let $\gamma' \in L$ be s.th. $f \in \mathrm{im}(\psi_{\gamma'})$. Let further $f_{\gamma'} \in B_{\gamma'}$ with $\psi_{\gamma'}(f_{\gamma'})$. Since L is a directed set, there is a upper bound $\hat{\gamma} \in L$ with $\gamma, \gamma' \leq \hat{\gamma}$. We then have:

$$\tilde{\psi}_\gamma(f_\gamma) = \tilde{\psi}_{\hat{\gamma}}(\psi_{\hat{\gamma}\gamma} f_\gamma) = \tilde{\psi}_{\hat{\gamma}}(\psi_{\hat{\gamma}\gamma'} f_{\gamma'}) = \tilde{\psi}_{\gamma'}(f_{\gamma'})$$

That it is a linear isometry follows, since ψ_γ and $\tilde{\psi}_\gamma$ are linear isometries. Hence it is also bounded and hence we can define u via the unique extension using BLT.
We now prove, that u is unique. Therefore it is sufficient, that it is unique on $\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$. Assume we have a map $u' : \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma) \to Y$ with $u' \neq u$. Then there must be a $f \in \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$ such that $u(f) \neq u'(f)$. But we now have by commutativity, that for all $\gamma \in L$: $u'(f) = \tilde{\psi}_\gamma(\psi^{-1}(f)) = u(f)$ and hence u is unique on $\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$ and hence unique on $\lim_\to X_\gamma$.

Hence the inductive limit in the category of Banach spaces is given by the tuple $(\lim_\to B_\gamma, \psi_\gamma)$ as defined above. Uniqueness follows with theorem 9. □

We now show the following Lemma, which is quite important for further applications:

Theorem 11 (cp. Ex. 11.5.26 of [18] for a Hilbert space version)
Let $(B_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive family of Banach spaces and $(\lim_\to B_\gamma, \psi_\gamma)$ be its inductive limit. Let A be a Banach space together with maps $\tilde{\psi}_\gamma : B_\gamma \to A$ s.th.

1. *$\forall \gamma \in L : \tilde{\psi}_\gamma$ is an isometry.*
2. *$\forall \gamma' \geq \gamma \in L : \tilde{\psi}_{\gamma'} \circ \psi_{\gamma'\gamma} = \tilde{\psi}_\gamma$.*
3. *$\bigcup_{\gamma\in L} \mathrm{im}(\tilde{\psi}_\gamma)$ is dense in A.*

Then the universal map $u : \lim_\to B_\gamma \to A$ is a bijective isometry. Hence $\lim_\to B_\gamma$ and A are isomorphic as Banach spaces.

Proof. Please recall first, how the universal map is constructed.
Let $f \in \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$. Then $\exists \gamma \in L : \exists_1 f_\gamma \in B_\gamma : \psi_\gamma(f_\gamma) = f$ and we set for those $\gamma \in L$: $\psi_\gamma^{-1}(f) = f_\gamma$. Now define the universal map as $u(f) = \tilde{\psi}_\gamma \circ \psi_\gamma^{-1}(f)$.
Now we want to show, that this u defines a bijective isometry

$$u : \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma) \to \bigcup_{\gamma\in L} \mathrm{im}(\tilde{\psi}_\gamma)$$

, since then it extends by lemma 6 uniquely to an bijective isometry $\tilde{u} : \lim_{\rightarrow} B_\gamma \to A$.

First observe, that u is a composition of isometries and hence an isometry. Hence it is also injective. Now towards surjectivity: Let $g \in \bigcup_{\gamma \in L} \operatorname{im}(\tilde{\psi}_\gamma)$. Then there is $\gamma \in L$ with $g_\gamma \in B_\gamma$ such that $g = \tilde{\psi}_\gamma g_\gamma$. We further have $\psi_\gamma(g_\gamma) \in \operatorname{im}(\psi_\gamma) \subset \bigcup_{\gamma \in L} \operatorname{im}(\psi_\gamma)$. Further we have $u(\psi_\gamma(g_\gamma)) = g$ by construction of the universal map and hence the assertion follows. Now the claim follows by lemma 6. $\square$

We now want to show the following easy corollar. On the one hand it provides a nice example, and on the other hand we will need it later in our discussion on polymer and loop quantization:

Corollar 3
Let B be a Banach space, L be a label set and $(B_\gamma)_{\gamma \in L}$ be a family of Banach spaces with:

1. *$\forall \gamma \in L : B_\gamma \subset B$ is a linear subspace.*
2. *$\forall \gamma \leq \gamma' \in L : B_\gamma \subset B_{\gamma'}$.*

Then:

1. *$(B_\gamma, i_{\gamma\gamma'})_{\gamma' \geq \gamma \in L}$, with $i_{\gamma\gamma'} : B_\gamma \hookrightarrow B_{\gamma'}$ being the canonical injection, is an inductive system in the category of Banach spaces.*
2. *The inductive limit of this inductive family is given by:*

$$\lim_{\rightarrow} B_\gamma = \overline{\bigcup_{\gamma \in L} B_\gamma} \subset B$$

together with the obvious maps $i_\gamma : B_\gamma \to \lim_{\rightarrow} B_\gamma$.

Proof. (1) The claim follows directly, since each $i_{\gamma'\gamma}$ is an isometry by definition and we have also for $\gamma'' \geq \gamma' \geq \gamma \in L$ directly:

$$\begin{aligned} i_{\gamma''\gamma'} \circ i_{\gamma'\gamma} &= i_{\gamma''\gamma} \\ i_{\gamma\gamma} &= id. \end{aligned}$$

(2) The canonically defined map

$$i_\gamma : B_\gamma \hookrightarrow \bigcup_{\gamma \in L} B_\gamma \subset B$$

is an isometry and hence also injective. Further we have, that $\bigcup_{\gamma \in L} \operatorname{im}(i_\gamma) = \bigcup_{\gamma \in L} B_\gamma$ is dense in $\overline{\bigcup_{\gamma \in L} B_\gamma}$. We have also directly, that

$$i_{\gamma'} \circ i_{\gamma'\gamma} = i_\gamma$$

for all $\gamma' \geq \gamma \in L$ and hence the claim follows with theorem 11. $\square$

3.1.3. Projective Limits in the Category of Compact Hausdorff Spaces

We show the following theorem:

Theorem 12 (cp. [27] for 1.) and [17] for 2.) with $L = \mathbb{N}$)
Let $\underline{Top}$ be the category whose objects are topological spaces and whose morphisms are continuous surjections. Further let $\underline{CH}$ be the category whose objects are compact Hausdorff spaces and whose morphisms are continuous surjections. Then:

1. *The projective limit exists in $\underline{Top}$ and is unique.*

2. *The projective limit exists in $\underline{CH}$ and is unique.*

Proof. 1.) Let $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ be a projective family of topological spaces. We first construct the conjectured projective limit as

$$\lim_{\leftarrow} X_\gamma := \left\{ (x_\gamma)_{\gamma\in L} \in \prod_{\gamma\in L} X_\gamma \middle| \forall\gamma' \geq \gamma : p_{\gamma\gamma'}x_{\gamma'} = x_\gamma \right\} \subset \prod_{\gamma\in L} X_\gamma$$

topologized by the subspace topology of the product topology. We further define maps $p_\gamma : \lim_{\leftarrow} X_\gamma \to X_\gamma$ as

$$p_\gamma = \pi_\gamma \circ i$$

, where $\pi_\gamma : \prod_{\gamma\in L} X_\gamma \to X_\gamma$ is the canonical projection and $i : \lim_{\leftarrow} X_\gamma \hookrightarrow \prod_{\gamma\in L} X_\gamma$ is the canonical embedding. Continuity of p_γ is clear, since π_γ and i are continuous. Further $(p_\gamma)_{\gamma\in L}$ satisfies the composition property of the projective limit, since for $(x_\gamma)_{\gamma\in L} \in \lim_{\leftarrow} X_\gamma$ we have $p_{\gamma\gamma'}x_{\gamma'} = x_\gamma$ and further $p_\gamma\left((x_\gamma)_{\gamma\in L}\right) = x_\gamma$ and $p_{\gamma'}\left((x_\gamma)_{\gamma\in L}\right) = x_{\gamma'}$, which gives $p_{\gamma\gamma'} \circ p_{\gamma'} = p_\gamma$.

We now show the universal property: Let Y be a topological space and let $(\tilde{p}_\gamma)_{\gamma\in L})$ be a family of projections with:

$$\forall\gamma' \geq \gamma : \tilde{p}_\gamma \circ p_{\gamma\gamma'} = \tilde{p}_{\gamma'}$$

Then construct a map

$$u : Y \to \lim_{\leftarrow} X_\gamma, y \mapsto (\tilde{p}_\gamma(y))_{\gamma\in L}$$

We now have to show, that this map is well defined and continuous.

We first show well definedness: Let $y \in Y$. Now $p_{\gamma\gamma'} \circ \tilde{p}_{\gamma'}(y) = \tilde{p}_\gamma(y)$ by above. Hence we have $u(y) \in \lim_{\leftarrow} X_\gamma$.

Now continuity: Since the topology on $\lim_{\leftarrow} X_\gamma$ is given by the subspace topology of the product topology, u is continuous, iff for all $\gamma \in L$ it holds, that the map $\psi_\gamma \circ i \circ u : Y \to X_\gamma$ is continuous. Now we have, that $\pi_\gamma \circ i = p_\gamma$ and hence, we have to show, that $\forall\gamma \in L : p_\gamma \circ u$ is continuous. But now $p_\gamma \circ u = \tilde{p}_\gamma$ holds and hence $p_\gamma \circ u$ is continuous, since $\tilde{p}_\gamma$ is.

Uniqueness follows with theorem 9.

2.) We now want to show, that $\lim_{\rightarrow} X_\gamma$ is a compact Hausdorff space, if the X_γ are compact Hausdorff spaces.

First we have, that arbitrary products of Hausdorff spaces are Hausdorff, and subspaces of Hausdorff spaces are also Hausdorff. We hence have, that $\lim_{\leftarrow} X_\gamma \subseteq \prod_{\gamma\in L} X_\gamma$ is Hausdorff.

Further we have the following fact: Let Y be a compact Hausdorff space and $X \subset Y$. Then X is compact iff X is closed in Y. Therefore we want to show, that $\lim_{\leftarrow} X_\gamma$ is closed in $\prod_{\gamma\in L} X_\gamma$, since then compactness follows. Therefore define for $\gamma, \gamma' \in L$ with $\gamma \leq \gamma'$ the following set:

$$C(\gamma', \gamma) = \left\{ (x_\gamma)_{\gamma\in L} \in \prod_{\gamma\in L} X_\gamma \middle| p_{\gamma\gamma'}x_{\gamma'} = x_\gamma \right\}$$

We now show:

Assertion 1: $\lim_{\leftarrow} X_\gamma = \bigcap_{\gamma' \geq \gamma \in L} C(\gamma', \gamma)$.

Proof of Assertion 1: We have:

$$\lim_{\leftarrow} X_\gamma = \left\{ (x_\gamma)_{\gamma \in L} \in \prod_{\gamma \in L} X_\gamma \middle| \forall \gamma' \geq \gamma \in L : p_{\gamma\gamma'} x_{\gamma'} = x_\gamma \right\}$$

Further we have by the definition of the intersection:

$$\bigcap_{\gamma' \geq \gamma \in L} C(\gamma', \gamma) = \left\{ (x_\gamma)_{\gamma \in L} \in \prod_{\gamma \in L} X_\gamma \middle| \forall \gamma' \geq \gamma \in L : p_{\gamma\gamma'} x_{\gamma'} = x_\gamma \right\}$$

and hence both sets are equal.

Assertion 2: $C(\gamma', \gamma)$ is closed for all $\gamma' \geq \gamma \in L$.

Proof of Assertion 2: Let $x \in \left(\prod_{\gamma \in L} X_\gamma \right) \setminus C(\gamma', \gamma)$, i.e.:

$$(x_\gamma)_{\gamma \in L} \in \prod_{\gamma \in L} X_\gamma \wedge p_{\gamma\gamma'} x_{\gamma'} \neq x_\gamma$$

By the Hausdorff property, we hence have two neighborhoods

$$x_\gamma \in U \subset X_\gamma$$
$$p_{\gamma\gamma'} x_{\gamma'} \in \tilde{V} \subset X_\gamma$$

such that $U \cap \tilde{V} = \emptyset$ since $p_{\gamma\gamma'} x_{\gamma'} \neq x_\gamma$.

With this we obtain an open set $V \subset X_{\gamma'}$ via

$$\forall x_{\gamma'} \in V : p_{\gamma\gamma'}^{-1}(\tilde{V}) \subset X_{\gamma'}$$

which has the property

$$\forall x_{\gamma'} \in V : p_{\gamma\gamma'} x_{\gamma'} \notin U$$

, since $p_{\gamma\gamma'}(V) \cap U = \tilde{V} \cap U = \emptyset$.

Now let for $\alpha \in L \setminus \{\gamma, \gamma'\}$ $U_\alpha \subset X_\alpha$ be an arbitrary open neighborhood of x_α and set further $U_\gamma = U$ and $U_{\gamma'} = V$. Now define

$$O = \prod_{\alpha \in L} U_\alpha$$

, which is open as a product of open sets. Further $\forall \alpha \in L : x_\alpha \in U_\alpha$ by construction, and hence it is a neighborhood of x.

Now let $(y_\gamma)_{\gamma \in L} \in O$. Especially we have $y_\gamma \in U$ and $y_{\gamma'} \in V$. Now by above argumentation we have:

$$p_{\gamma\gamma'} y_{\gamma'} \neq y_\gamma \Rightarrow (y_\gamma)_{\gamma \in L} \in \left(\prod_{\gamma \in L} X_\gamma \right) \setminus C(\gamma', \gamma)$$

Hence $\left(\prod_{\gamma \in L} X_\gamma \right) \setminus C(\gamma', \gamma)$ is open, and hence the claim of the theorem follows, since arbitrary intersections of closed sets are closed. □

Further we show the following Lemma for later use:

Lemma 7
Let $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ be an projective family of topological spaces. Let X be a topological space. Let $f : X \to \lim_{\leftarrow} X_\gamma$ be a map. Then: f is continuous if and only if $p_\gamma \circ f$ is continuous for all $\gamma \in L$

Proof. We have topologized the projective limit via the subspace topology of the product topology. Hence we have, that $f : X \to \lim_{\leftarrow} X_\gamma$ is continuous if and only if $i \circ f : X \to \prod_{\gamma \in L} X_\gamma$ is continuous, where $i : \lim_{\leftarrow} X_\gamma \hookrightarrow \prod_{\gamma \in L} X_\gamma$ is the canonical injection. Further we have, that this map is continuous if and only if for all $\gamma \in L$ we have, that $\pi_\gamma \circ i \circ f : X \to X_\gamma$ is continuous, where $\pi_\gamma : \prod_{\gamma \in L} X_\gamma \to X_\gamma$ is the canonical projection. Since $p_\gamma = \pi_\gamma \circ i$, the claim follows. □

3.2. Inductive Limits of Hilbert Spaces I: Hilbert Spaces and Operators

In this section we will prove the existence of inductive limits in the category of Hilbert spaces. Further we will define the notion of inductive families of bounded and unbounded operators, define the notion of inductive limits thereof and show their existence.

We first recall the following theorem:

Lemma 8 (Parallelogram law, cp. Thm. V.1.7 of [36])
Let $(X, \|\cdot\|)$ be a Banach space. Then the norm $\|\cdot\|$ is induced by an inner product if and only if the parallelogram law

$$\forall x, y \in X : \|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

holds. In this case the inner product that induces $\|\cdot\|$ is given by the polarization identity:

$$\forall x, y \in X : \langle x, y\rangle = \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2 - i\|y+ix\|^2 + i\|y-ix\|^2\right)$$

Now we show, that the inductive limit of Hilbert spaces exists and is unique:

Theorem 13 (Inductive limit exists in the Category of Hilbert spaces, cp. Ex. 11.5.26 of [18])
Let $\underline{Hil}$ be the category whose objects are Hilbert spaces and whose morphisms are linear isometries. Then the inductive limit exists in this category and is unique.

Proof. Let $(\mathcal{H}_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ be an inductive family of Hilbert spaces. Then let $(\lim_{\rightarrow} \mathcal{H}_\gamma, (\psi_\gamma)_{\gamma \in L})$ be the inductive limit of $(\mathcal{H}_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ as a inductive family of Banach spaces. We now show, that the norm on $\lim_{\rightarrow} \mathcal{H}_\gamma$ satisfies the parallelogram law. We do this in two steps. First we show, that the parallelogram law is satisfied on $\bigcup_{\gamma \in L} \operatorname{im}(\psi_\gamma)$. Then we show, that the parallelogram law holds on a Banach space, if it is satisfied on a dense subspace.

First consider $f, g \in \bigcup_{\gamma \in L} \operatorname{im}(\psi_\gamma)$. Hence we have $\gamma_1, \gamma_2 \in \mathcal{L}$ and $f_{\gamma_1} \in \mathcal{H}_{\gamma_1}$, $g_{\gamma_2} \in \mathcal{H}_{\gamma_2}$ such that $f = \psi_{\gamma_1}(f_{\gamma_1})$ and $g = \psi_{\gamma_2}(g_{\gamma_2})$. Now, since L is a label set we have a $\hat{\gamma} \geq \gamma_1, \gamma_2$ and further we have:

$$f = \psi_{\gamma_1}(f_{\gamma_1}) = \psi_{\hat{\gamma}}\psi_{\hat{\gamma}\gamma_1}(f_{\gamma_1}) = \psi_{\hat{\gamma}}(f_{\hat{\gamma}})$$
$$g = \psi_{\gamma_2}(g_{\gamma_2}) = \psi_{\hat{\gamma}}\psi_{\hat{\gamma}\gamma_2}(g_{\gamma_2}) = \psi_{\hat{\gamma}}(g_{\hat{\gamma}})$$

with $f_{\hat{\gamma}} := \psi_{\hat{\gamma}\gamma_1} f_{\gamma_1}$ and $g_{\hat{\gamma}} := \psi_{\hat{\gamma}\gamma_2} g_{\gamma_2}$. We now have:

$$\begin{aligned} \|f+g\|^2 + \|f-g\|^2 &= \|\psi_{\hat{\gamma}}(f_{\hat{\gamma}} + g_{\hat{\gamma}})\| + \|\psi_{\hat{\gamma}}(f_{\hat{\gamma}} - g_{\hat{\gamma}})\| \\ &= \|f_{\hat{\gamma}} + g_{\hat{\gamma}}\|_{\hat{\gamma}} + \|f_{\hat{\gamma}} - g_{\hat{\gamma}}\|_{\hat{\gamma}} \\ &= 2\left(\|f_{\hat{\gamma}}\|_{\hat{\gamma}}^2 + \|g_{\hat{\gamma}}\|_{\hat{\gamma}}^2\right) \\ &= 2\left(\|\psi_{\hat{\gamma}}(f_{\hat{\gamma}})\|^2 + \|\psi_{\hat{\gamma}}(g_{\hat{\gamma}})\|^2\right) \\ &= 2\left(\|f\|^2 + \|g\|^2\right) \end{aligned}$$

Where we have used, that the parallelogram law holds on $\mathcal{H}_{\hat{\gamma}}$ and that $\psi_{\hat{\gamma}}$ is an isometry.

We now show, that if B is a Banach space and $A \subset B$ is a dense subset on which the parallelogram law holds, then the parallelogram law holds on B.

Therefore let $f, g \in B$ and $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subset A$ be sequences with $f_n \to f$ and $g_n \to g$ for $n \to \infty$. We then have directly:

$$\|f_n + g_n\|^2 + \|f_n - g_n\|^2 = 2\left(\|f_n\|^2 + \|g_n\|^2\right)$$

and hence

$$\begin{aligned} \|f+g\|^2 + \|f-g\|^2 &= \lim_{n\to\infty}\left(\|f_n + g_n\|^2 + \|f_n - g_n\|^2\right) \\ &= \lim_{n\to\infty} 2\left(\|f_n\|^2 + \|g_n\|^2\right) \\ &= 2\left(\|f\|^2 + \|g\|^2\right) \end{aligned}$$

and hence the claim follows.

With this the assertion is proven. □

We now want to understand, how orthonormal bases behave under inductive limits:

Theorem 14 (Inductive limit of orthonormal bases)
Let $(\mathcal{H}_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive system of Hilbert spaces and let $(S_\gamma)_{\gamma\in L}$ be a family of orthonormal bases with:

1. *For each $\gamma \in L$ it holds, that S_γ is an orthonormal base of $\mathcal{H}_\gamma$.*
2. *For each $\gamma' \geq \gamma \in L$ we have, that $\psi_{\gamma'\gamma} S_\gamma \subset S_{\gamma'}$.*

Then:

$$S := \bigcup_{\gamma\in L} \psi_\gamma(S_\gamma)$$

is an orthonormal base for $\lim_{\to} \mathcal{H}_\gamma$.

Proof. We first show, that S is an orthonormal system, i.e. $\forall e_i \in S : \|e_i\| = 1$ and $\forall e_i \neq e_j \in S : \langle e_i, e_j\rangle = 0$ (cp. Def. V.4.1 of [36]). This is quite easy to show:

Let $e_i \in S$. Then there is a $\gamma \in L$ and a $e_i^{(\gamma)} \in S_\gamma \subset \mathcal{H}_\gamma$ with $\psi_\gamma(e_i^{(\gamma)}) = e_i$. Since S_γ is orthonormal base and ψ_γ is an isometry, we have $\|e_i\| = \|\psi_\gamma(e_i^{(\gamma)})\| = \|e_i^{(\gamma)}\|_\gamma = 1$.

Now let $e_i, e_j \in S$. Then there are $\gamma_1, \gamma_2 \in L$ and $e_i^{(\gamma_1)} \in S_{\gamma_1}$, $e_j^{(\gamma_2)} \in S_{\gamma_2}$ with $\psi_{\gamma_1}(e_i^{(\gamma_1)}) = e_i$ and $\psi_{\gamma_2}(e_j^{\gamma_2}) = e_j$. Now there is, since L is a label set, a $\hat{\gamma}$ with $\hat{\gamma} \geq \gamma_1, \gamma_2$. With this we have:

$$\begin{aligned} e_i &= \psi_{\gamma_1}(e_i^{(\gamma_1)}) = \psi_{\hat{\gamma}}\psi_{\hat{\gamma}\gamma_1} e_i^{(\gamma_1)} = \psi_{\hat{\gamma}} e_i^{(\hat{\gamma})} \\ e_j &= \psi_{\gamma_2}(e_j^{(\gamma_2)}) = \psi_{\hat{\gamma}}\psi_{\hat{\gamma}\gamma_2} e_j^{(\gamma_2)} = \psi_{\hat{\gamma}} e_j^{(\hat{\gamma})} \end{aligned}$$

With $e_i^{(\hat{\gamma})} = \psi_{\hat{\gamma}\gamma_1} e_i^{(\gamma_1)}$ and $e_j^{\hat{\gamma}} = \psi_{\hat{\gamma}\gamma_2} e_j^{\gamma_2}$. We then have since $\psi_{\hat{\gamma}}$ is isometry and $S_{\hat{\gamma}}$ is orthonormal base, that:

$$\begin{aligned}\langle e_i, e_j \rangle &= \langle \psi_{\hat{\gamma}}(e_i^{(\hat{\gamma})}), \psi_{\hat{\gamma}}(e_j^{(\hat{\gamma})}) \rangle \\ &= \langle e_i^{(\hat{\gamma})}, e_j^{(\hat{\gamma})} \rangle_{\hat{\gamma}} \\ &= 0\end{aligned}$$

And hence we have shown, that S is an orthonormal system.

Now the following holds: An orthonormal system S of an Hilbertspace $\mathcal{H}$ is an orthonormal basis if and only if $\overline{\text{lin}} S = \mathcal{H}$ (cp. Thm V.4.9 of [36]). Hence we show, that $\text{lin}S$ is dense in $\lim_{\rightarrow} \mathcal{H}_\gamma$.

Therefore we first show, that $\text{lin}(S) \subset \bigcup_{\gamma \in L} \text{im}(\psi_\gamma)$ is dense in the latter. Therefore let $f \in \bigcup_{\gamma \in L} \text{im}(\psi_\gamma)$. I.e. we have a $\gamma \in L$ and a $f_\gamma \in \mathcal{H}_\gamma$ with $\psi_\gamma(f_\gamma) = f$. Now we have a sequence $(e_n^\gamma)_{n \in \mathbb{N}} \in \text{lin}(S_\gamma)$ with $e_n^\gamma \rightarrow f_\gamma$ since S_γ is ONB for $\mathcal{H}_\gamma$. With this we have:

$$\begin{aligned}\|\psi_\gamma(e_n^\gamma) - f\| &= \|\psi_\gamma(e_n^\gamma - f_\gamma)\| \\ &= \|e_n^\gamma - f_\gamma\|_\gamma < \epsilon\end{aligned}$$

and hence $\psi_\gamma(e_n^\gamma) \rightarrow f$ in $\bigcup_{\gamma \in L} \text{im}(\psi_\gamma)$. Further we have that $(\psi_\gamma(e_n^\gamma))_{n \in \mathbb{N}} \subset \psi_\gamma(\text{lin}S_\gamma) = \text{lin}\psi_\gamma(S_\gamma) \subset \text{lin}(S)$. And hence the claim follows, since denseness is transitive. □

For later use we show the following theorem:

Corollar 4 (cp. Ex. 11.5.26 of [18])
Let $(\mathcal{H}_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ be an inductive family of Hilbert spaces and $(\lim_{\rightarrow} \mathcal{H}_\gamma, \psi_\gamma)$ be its inductive limit. Let $\mathcal{K}$ be a Hilbert space together with linear maps $\tilde{\psi}_\gamma : \mathcal{H}_\gamma \rightarrow \mathcal{K}$ s.th.

1. *$\forall \gamma \in L : \tilde{\psi}_\gamma$ is an isometry.*
2. *$\forall \gamma' \geq \gamma \in L : \tilde{\psi}_{\gamma'} \circ \psi_{\gamma'\gamma} = \tilde{\psi}_\gamma$.*
3. *$\bigcup_{\gamma \in L} \text{im}(\tilde{\psi}_\gamma)$ is dense in $\mathcal{K}$.*

Then the universal map $u : \lim_{\rightarrow} \mathcal{H}_\gamma \rightarrow \mathcal{K}$ is a bijective isometry. Hence $\lim_{\rightarrow} \mathcal{H}_\gamma$ and $\mathcal{K}$ are unitary equivalent.

Proof. This follows directly with theorem 11 and by the fact that bijective isometries between Hilbert spaces are automatically unitary. □

We now want to introduce the notion of an inductive family of operators:

Definition 16 (Inductive system and inductive limit of bounded operators, cp. Ex. 11.5.27 of [18])
Let $(B_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ be an inductive system of Banach spaces. Then:

1. *An inductive family of bounded operators is a family $(T_\gamma)_{\gamma \in L}$ with*
   a) *$\forall \gamma \in L : T_\gamma \in \mathcal{B}(B_\gamma)$*
   b) *$\sup_{\gamma \in L} \|T_\gamma\| < \infty$*
   c) *There is a $\gamma_0 \in L$ such that $\forall \gamma' \geq \gamma \geq \gamma_0$ it holds, that $T'_\gamma \circ \psi_{\gamma'\gamma} = \psi_{\gamma'\gamma} \circ T_\gamma$.*
2. *The inductive limit of above inductive family of operators is an operator $\lim_{\rightarrow} T_\gamma \in \mathcal{B}(\lim_{\rightarrow} B_\gamma)$ such that for all $\gamma \geq \gamma_0$ it holds, that $T \circ \psi_\gamma = \psi_\gamma \circ T_\gamma$.*

One then can show:

Theorem 15 (cp. Ex. 11.5.27 of [18])
Let $(T_\gamma)_{\gamma\geq\gamma_0\in L}$ be an inductive family of bounded operators on an inductive family of Banach spaces $(B_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$. Then its inductive limit exists and is unique.

Proof. We define the conjectured inductive limit on $\bigcup_{\gamma\in L}\mathrm{im}(\psi_\gamma)$ as follows: Let $f \in \bigcup_{\gamma\in L}\mathrm{im}(\psi_\gamma)$. Then set:

$$(\lim_{\to} T_\gamma)f = \begin{cases} \psi_\gamma T_\gamma \psi_\gamma^{-1}(f) & \text{if} \gamma \geq \gamma_0 \\ 0 & \text{else} \end{cases}$$

That this is independent of the choice of γ follows from the fact, that L is directed by the usual discussion. We now have to show, that this defines a bounded operator, that it is an inductive limit and that the inductive limit is unique.

We first show, that $\lim_\to T_\gamma \in \mathcal{B}(\lim_{\to B_\gamma})$. Therefore recall, that the ψ_γ are isometries. Hence we have for $f \in \mathrm{im}(\psi_\gamma)$ with $\gamma \geq \gamma_0$:

$$\|\lim_{\to} T_\gamma f\|_{\lim_\to B_\gamma} = \|\psi_\gamma T_\gamma \psi_\gamma^{-1} f\|_\gamma = \|T_\gamma \psi_\gamma^{-1}\|_\gamma$$

and $\|\lim_\to T_\gamma f\|_{\lim_\to B_\gamma} = 0$ otherwise. Hence we have:

$$\begin{aligned}\|T_\gamma\| &\leq \sup_{\gamma\in L}\sup_{f_\gamma\in B_\gamma} \frac{\|T_\gamma f_\gamma\|_\gamma}{\|f_\gamma\|_\gamma} \\ &= \sup_{\gamma\in L}\|T_\gamma\| < \infty\end{aligned}$$

Hence its unique extension exists and is bounded by BLT.

Further $\lim_\to T_\gamma$ is really an inductive limit, since:

$$(\lim_{\to} T_\gamma)\circ\psi_\gamma = \psi_\gamma T_\gamma \psi_\gamma^{-1}\circ\psi_\gamma = \psi_\gamma T_\gamma$$

Now assume, that there is another bounded operator R on $\lim_\to B_\gamma$ with $R\psi_\gamma = \psi_\gamma T_\gamma$ and $R \neq \lim_\to T_\gamma$. By $R\psi_\gamma = \psi_\gamma T_\gamma$ we have, that $R = \lim_\to T_\gamma$ on $\bigcup_{\gamma\in L}\mathrm{im}\psi_\gamma$ since:

$$\forall f \in \mathrm{im}(\psi_\gamma) : Rf = R\phi_\gamma a_\gamma = \phi_\gamma T_\gamma f_\gamma = T\phi_\gamma f_\gamma = Ta$$

Where we have set $f_\gamma = \psi_\gamma^{-1}(f)$. But by BLT we would then have $R = \lim_\to T_\gamma$.

Hence the assertion follows. □

3.3. Inductive Limits of C^*-Algebras I: C^*-Algebras and their Spectra

We need first versions of lemma 5 and lemma 6 for C^*-algebras.

Lemma 9 (BLT for C^*-algebras)
Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be C^*-algebras and let $V_1 \subset \mathfrak{U}_1$ be a dense $*$-subalgebra. Let further $\phi : V_1 \to \mathfrak{U}_2$ be a $*$-morphism.
Then:

1. *There is a unique $*$-morphism $\tilde{\phi} : \mathfrak{U}_1 \to \mathfrak{U}_2$ which extends ϕ.*

2. *Further $\tilde{\phi}$ is an isometry if ϕ is an isometry.*

3. *Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be unital. Let further the unit of $\mathfrak{U}_1$ be contained in V_1. Let ϕ be unital. Then $\tilde{\phi}$ is unital.*

Proof. Let $a, b \in \mathfrak{U}_1$ and $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subset V_1$ with $a_n \to a$ and $b_n \to b$.
Then observe first, that $a_n b_n \to ab$ holds, since:

$$\begin{aligned} 0 &\leq \|a_n b_n - ab\| \\ &= \|a_n b_n - a_n b + a_n b - ab\| \\ &\leq \|a_n b_n - a_n b\| + \|a_n b - ab\| \\ &\leq \|a_n\| \|b_n - b\| + \|b\| \|a_n - a\| \\ &\leq \left(\sup_{n\in\mathbb{N}} \|a_n\|\right) \|b_n - b\| + \|b\| \|a_n - a\| \to 0 \end{aligned}$$

And further $a_n^* \to a^*$ holds, since:

$$0 \leq \|a_n^* - a^*\| = \|(a_n - a)^*\| = \|a_n - a\| \to 0$$

Hence we have

$$\begin{aligned} \lim_{n\to\infty} a_n b_n &= \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) \\ \lim_{n\to\infty} a_n^* &= \left(\lim_{n\to\infty} a_n\right)^* \end{aligned}$$

We now have with lemma 5, that the unique extension of ϕ s defined via $\tilde{\phi} : \mathfrak{U}_1 \to \mathfrak{U}_2, c \mapsto \tilde{\phi}c = \lim_{n\to\infty} \phi c_n$ for a sequence $(c_n)_{n\in\mathbb{N}} \subset V_1$ with $\lim_{n\to\infty} c_n = c$. Further it is independent of the choice of the sequence. We now show, that $\tilde{\phi}$ is a $*$-morphism. Therefore observe:

$$\tilde{\phi}(ab) = \lim_{n\to\infty} \tilde{\phi}(a_n b_n) = \lim_{n\to\infty} (\phi a_n)(\phi b_n) = \left(\lim_{n\to\infty} \phi a_n\right)\left(\lim_{n\to\infty} \phi b_n\right) = (\tilde{\phi} a)(\tilde{\phi} b)$$

and

$$(\tilde{\phi} a^*) = \lim_{n\to\infty} (\phi a_n^*) = \lim_{n\to\infty} (\phi a_n)^* = \tilde{\phi}(a)^*$$

2.) We know from lemma 5, that this holds.
3.)We have, that $\tilde{\phi}\Big|_{V_1} = \phi$ holds. Further the identity of $\mathfrak{U}_1$ is contained in V_1 and ϕ is unital. Hence the assertion follows directly. □

We now want to show a version of lemma 6 for C^*-algebras:

Lemma 10
Let $\mathfrak{U}_1$, $\mathfrak{U}_2$ be C^*-algebras and let $V_1 \subset \mathfrak{U}_1$, $V_2 \subset \mathfrak{U}_2$ be dense $*$-subalgebras. Let further $\phi : V_1 \to V_2$ be a bounded, isometric. $*$-isomorphism. Then the unique bounded extension $\tilde{\phi} : \mathfrak{U}_1 \to \mathfrak{U}_2$ from the lemma 10 is an isometric $*$-isomorphism. Further, if the unit of $\mathfrak{U}_1$ is contained in V_1 and ϕ is unital, then $\tilde{\phi}$ is unital.

Proof. It follows by lemma 6, that $\tilde{\phi}$ is a bijective isometry. Further it follows by lemma 9, that it is a $*$-morphism. Further it follows by lemma 9, that it is unital in the unital case. Hence the claim follows. □

We now want to show, that the inductive limit exists in the category of C^*-Algebras:

Theorem 16 (cp. App. L of [35])
Let $\underline{Cst}$ be the category, whose objects are C^*-algebras and whose morphisms are isometric $*$-morphisms. Then the inductive limit exists in this category and is unique. Let further $\underline{Cst_{\mathbb{1}}}$ be the category, whose objects are unital C^*-algebras and whose morphisms are unital isometric $*$-morphisms. Then the inductive limit exists also in this category and is unique. Finally the inductive limit of an inductive family of abelian C^*-algebras is abelian.

Proof. Let $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma',\gamma\in L}$ be an inductive family of C^*-algebras. Now let $(\lim_\rightarrow \mathfrak{U}_\gamma, (\phi_\gamma)_{\gamma\in L})$ be the inductive limit of this inductive family in the category of Banach spaces.We now want to show, that this defines also an inductive limit in the categories $\underline{Cst}$ and $\underline{Cst_{\mathbb{1}}}$.

Assertion 1: $\mathfrak{X}$ from the proof of theorem 10 is a C^*-algebra w.r.t. the pointwise $*$-algebra structure and the sup norm. In the case of $\underline{Cst_{\mathbb{1}}}$ it is further unital.

Proof of Assertion 1: Since $\mathfrak{X}$ is a Banach space, it remains to show, that it defines a normed $*$-algebra satisfying the C^*-property. We therefore define a $*$-algebra structure on $\mathfrak{X}$ via

$$\forall (a_\gamma)_{\gamma\in L}, \forall (b_\gamma)_{\gamma\in L} \in \lim_{\rightarrow} \mathfrak{U}_\gamma : (a_\gamma)_{\gamma\in L}(b_\gamma)_{\gamma\in L} = (a_\gamma b_\gamma)_{\gamma\in L}$$
$$\forall (a_\gamma)_{\gamma\in L} \in \lim_{\rightarrow} \mathfrak{U}_\gamma : (a_\gamma)^*_{\gamma\in L} = (a^*_\gamma)_{\gamma\in L}$$

which is well defined, since

$$\sup_{\gamma\in L} \|a_\gamma b_\gamma\|_\gamma \leq \left(\sup_{\gamma\in L} \|a_\gamma\|_\gamma\right)\left(\sup_{\gamma\in L} \|b_\gamma\|_\gamma\right) < \infty$$
$$\sup_{\gamma\in L} \|x^*_\gamma\|_\gamma = \sup_{\gamma\in L} \|x_\gamma\|_\gamma < \infty$$

It can be easily checked, that this satisfies the property of a $*$-algebra. Further the C^*-property also follows directly, since:

$$\sup_{\gamma\in L} \|a^*_\gamma a_\gamma\|_\gamma = \sup_{\gamma\in L} \|a_\gamma\|^2 = \left(\sup_{\gamma\in L} \|a_\gamma\|\right)^2$$

In the unital case it can be easily checked, that $(\mathbb{1}_\gamma)_{\gamma\in L}$ is a unit for $\mathfrak{X}$, where $\mathbb{1}_\gamma$ are the units of the $\mathfrak{U}_\gamma$. Further

$$\sup_{\gamma\in L} \|\mathbb{1}_\gamma\|_\gamma = 1 < \infty$$

and hence $\mathbb{1}_\gamma \in \mathfrak{X}$.

Assertion 2: $\mathfrak{X}_0$ from the proof of theorem 10 is a norm-closed 2-sided ideal in $\mathfrak{X}$.

Proof of Assertion 2: That it defines a subalgebra can be shown easily, since it can be easily shown, that

$$\forall \alpha, \beta \in \mathbb{C} : \lim \|a_\gamma\|_\gamma = \lim \|b_\gamma\|_\gamma = 0 \Rightarrow \lim \|\alpha a_\gamma + \beta b_\gamma\|_\gamma = 0$$

, since $\|\alpha a_\gamma + \beta b_\gamma\|_\gamma \leq |\alpha|\|a_\gamma\|_\gamma + |\beta|\|b_\gamma\|_\gamma$ and further

$$\lim \|a_\gamma\|_\gamma = \lim \|b_\gamma\|_\gamma = 0 \Rightarrow \lim \|a_\gamma b_\gamma\|_\gamma = 0$$

, since $\|a_\gamma b_\gamma\|_\gamma \leq \|a_\gamma\|_\gamma \|b_\gamma\|_\gamma$ and finally

$$\lim \|a_\gamma\|_\gamma = 0 \Rightarrow \lim \|a_\gamma^*\|_\gamma = 0$$

, since $\|a_\gamma^*\| = \|a_\gamma\|$. Further it follows directly, that it defines a two-sided ideal. Therefore let $(a_\gamma) \in \mathfrak{X}$ and $(b_\gamma) \in \mathfrak{X}_0$. Then

$$0 \leq \|a_\gamma b_\gamma\|_\gamma \leq \|a_\gamma\|_\gamma \|b_\gamma\|_\gamma \leq \|a_\gamma\|_\gamma \|b_\gamma\|_\gamma \to 0$$

since $\|(a_\gamma)_{\gamma\in L}\| < \infty$. Hence it is a right ideal. That it is a left ideal follows analogously. That it is norm-closed follows was already shown in theorem 10.

Assertion 3: $\mathfrak{X}/\mathfrak{X}_0$ is a C^*-algebra and $\phi_\gamma := \psi_\gamma \mathfrak{U}_\gamma \to \mathfrak{X}/\mathfrak{X}_o$ as defined in assertion 4 of theorem 10 is a morphism in the category $\underline{Cst}$, i.e. an isometric $*$-morphism. Further for $\forall \gamma' \geq \gamma \in L$ it holds, that $\phi_{\gamma'} \circ \phi_{\gamma'\gamma} = \phi_\gamma$. Finally we have in the case of $\underline{Cst_{\mathbb{1}}}$, that $\mathfrak{X}/\mathfrak{X}_0$ and ϕ_γ are unital.

Proof of Assertion 3: That ϕ_γ is an isometry in the category $\underline{Ban}$ satisfying $\phi_{\gamma'} \circ \phi_{\gamma'\gamma} = \phi_\gamma$ was already shown in theorem 10. We now show, that it is a $*$-morphism. Therefore we first show, that j_γ is a $*$-morphism. This follows directly, since $\phi_{\gamma'\gamma}$ and the trivial map $a_\gamma \mapsto 0$ are $*$-morphisms. Hence $\phi_\gamma = \pi \circ j_\gamma$ is a $*$-morphism as a composition of $*$-morphisms. Now consider the unital case: That $\mathfrak{X}/\mathfrak{X}_0$ is a unital C^*-algebra follows directly with lemma 2. We now show, that ϕ_γ is unital. Therefore let $\mathbb{1}_\gamma \in \mathfrak{U}_\gamma$ be the unit element. We then have $j_\gamma(\mathbb{1}_\gamma) = (a_\beta)_{\beta\in L}$ with :

$$a_\beta = \begin{cases} \phi_{\beta\gamma}\mathbb{1}_\gamma & \beta \geq \gamma \\ 0 & \text{else} \end{cases}$$

Since $\phi_{\beta\gamma}$ is unital for all $\beta \geq \gamma \in L$, we have, that $\phi_{\beta\gamma}\mathbb{1}_\gamma = \mathbb{1}_\beta$. Let now $\mathbb{1} = (\mathbb{1}_\gamma)_{\gamma\in L}$ be the unit in $\mathfrak{X}$. Then:

$$(\mathbb{1} - (a_\beta)_{\beta\in L})_\beta = \begin{cases} 0 & \beta \geq \gamma \\ 1 & \text{else} \end{cases}$$

Hence the net $\Big(\| (\mathbb{1} - (a_\beta)_{\beta\in L})_\beta \|\Big)_{\beta\in L}$ converges to 0 and hence $\pi \circ j_\gamma(\mathbb{1}_\gamma) = \mathbb{1}$. Hence ϕ_γ is unital.

Assertion 4: $\mathrm{im}(\phi_\gamma) \subset \mathfrak{X}/\mathfrak{X}_0$ is a closed sub-algebra and $\forall \gamma' \geq \gamma : \mathrm{im}(\phi_\gamma) \subset \mathrm{im}(\phi_{\gamma'})$. In the case of $\underline{Cst_{\mathbb{1}}}$ $\mathrm{im}(\phi_\gamma)$ is unital.

Proof of assertion 4: Since $\mathfrak{U}_\gamma$ is closed C^*-algebra and ϕ_γ is an isometric $*$-morphism, we have, that $\mathrm{im}(\phi_\gamma)$ is closed subalgebra. That $\mathrm{im}(\phi_\gamma)$ is unital in the unital case, follows since ϕ_γ and $\mathfrak{U}_\gamma$ are unital. That $\mathrm{im}(\phi_\gamma) \subset \mathrm{im}(\phi_{\gamma'})$ for $\gamma' \geq \gamma$ was already shown in the proof of theorem 10.

Assertion 5 (basically a lemma): Let $\mathfrak{Z}$ be a C^*-algebra and let $\mathfrak{Z}_\gamma \subset \mathfrak{Z}$ be a family of C^*-subalgebras with $\forall \gamma' \geq \gamma : \mathfrak{Z}_\gamma \subseteq \mathfrak{Z}_{\gamma'}$. Then

$$\overline{\bigcup_{\gamma \in L} \mathfrak{Z}_\gamma} \subset \mathfrak{Z}$$

defines an C^*-algebra. Further, if $\mathfrak{Z}$ is unital and the $\mathfrak{Z}_\gamma$ are unital, then $\overline{\bigcup_{\gamma \in L} \mathfrak{Z}_\gamma}$ is also unital.

Proof of Assertion 5: We already know, that $\overline{\bigcup_{\gamma \in L} \mathfrak{Z}_\gamma}$ is a closed subspace and hence a Banach space. We now have to show, that it is closed under the $*$-algebraic structure: Let $b, a \in \overline{\bigcup_{\gamma \in L} \mathfrak{Z}_\gamma}$. Then there are sequences $(a_m)_{m \in \mathbb{N}}, (b_m)_{m \in \mathbb{N}} \subset \bigcup_{\gamma \in L} \mathfrak{Z}_\gamma$ with $a_m \to a$ and $b_m \to b$ for $m \to \infty$. Then:

$$\begin{aligned}
0 \leq \|a_m b_m - ab\| &= \|a_m b_m - a_m b + a_m b - ab\| \\
&\leq \|a_m b_m - a_m b\| + \|a_m b - ab\| \\
&\leq \|a_m\| \|b_m - b\| + \|a_m - a\| \|b\| \\
&\leq \left(\sup_{m \in \mathbb{N}} \|a_m\| \right) \|b_m - b\| + \|a_m - a\| \|b\| \\
&\to 0
\end{aligned}$$

and hence $a_m b_m \to ab$ for $m \to \infty$. Now we have to show, that $a_m b_m \in \bigcup_{\gamma \in L} \mathfrak{Z}_\gamma$ for all $m \in \mathbb{N}$. Since $a_m, b_m \in \bigcup_{\gamma \in L} \mathfrak{Z}_\gamma$ it follows that there are γ, γ' with $a_m \in \mathfrak{Z}_\gamma$ and $b_m \in \mathfrak{Z}_{\gamma'}$. Since L is a directed set, there is a $\hat{\gamma} \in L$ which is an upper bound for γ and γ' and hence $a_m, b_m \in \mathfrak{Z}_{\hat{\gamma}}$. Since the latter is a C^*-algebra, we have, that $a_m b_m \in \mathfrak{Z}_{\hat{\gamma}} \subset \bigcup_{\gamma \in L} \mathfrak{Z}_\gamma$. That it is closed under involution follows directly since $\|a_m^* - a^*\| = \|(a_m - a)^*\| = \|a_m - a\|$. The C^*-property follows, since $\bigcup_{\gamma \in L} \mathfrak{Z}_\gamma \subset \mathfrak{Z}$ and the latter is a C^*-algebra. In the unital case it holds, that $\mathfrak{Z}$ is unital and by uniqueness of the unit element hence we have, that the units of $\mathfrak{Z}$ and $\mathfrak{Z}_\gamma$ coincide. Hence $\mathbb{1}_{\mathfrak{Z}} \in \mathfrak{Z}_\gamma \subset \overline{\bigcup_{\gamma \in L} \mathfrak{Z}_\gamma} \subset \mathfrak{Z}$.

Assertion 6: The unique morphism u from the proof of theorem 10 is an isometric $*$-morphism. In the case of $\underline{Cst}_{\mathbb{1}}$ it is also unital.

Proof of Assertion 6: We have already shown in theorem 10, that u is unique, linear and isometric. We now show, that it defines a $*$-morphism. Therefore let $a, b \in \bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma)$. Hence we have a $\hat{\gamma} \in L$ with $a, b \in \mathrm{im}(\phi_{\hat{\gamma}})$. Then we have, that there are $a_{\hat{\gamma}}, b_{\hat{\gamma}} \in \mathfrak{U}_{\hat{\gamma}}$ with $\phi_{\hat{\gamma}}(a_{\hat{\gamma}}) = a$, $\phi_{\hat{\gamma}}(b_{\hat{\gamma}}) = b$ and further $\phi_{\hat{\gamma}}(a_{\hat{\gamma}} b_{\hat{\gamma}}) = ab$ since $\phi_{\hat{\gamma}}$ is $*$-morphism. Since it is further isometric and hence injective, we have that $\phi_{\hat{\gamma}}^{-1}(ab) = a_{\hat{\gamma}} b_{\hat{\gamma}}$. With the same argumentation we have $\phi_{\hat{\gamma}}^{-1}(a^*) = a_{\hat{\gamma}}^*$. Hence ϕ_γ^{-1} is a $*$-morphism on a dense subset. Hence u is defined as a composition of isometric $*$-morphisms on a dense $*$-subalgebra, and hence the claim follows with lemma 9. Finally it is unital in the unital case, since ϕ_γ is unital and hence $\phi_\gamma^{-1}(\mathbb{1}) = \mathbb{1}_\gamma$.

With this we have, that $\lim_\to \mathfrak{U}_\gamma := \overline{\bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma)}$ together with the maps ϕ_γ defines the inductive limit of the inductive system of C^*-algebras $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$. Uniqueness was discussed in theorem 10.

Now consider finally the abelian case. Therefore let $a, b \in \lim_\to \mathfrak{U}_\gamma = \overline{\bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma)}$. Hence $ab - ba \in \lim_\to \mathfrak{U}_\gamma$. Further there are sequences $(a_m)_{m \in \mathbb{N}}, (b_m)_{m \in \mathbb{N}} \subset \bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma)$ such that $a_m \to a$ and $b_m \to b$ for $m \to \infty$. Now we have, that $a_m b_m \to ab$ and

$b_m a_m \to ba$. Hence we have $a_m b_m - b_m a_m = [a_m, b_m] \to [a, b]$. Now observe, that for each a_m, b_m there is a $\hat{\gamma} \in L$ with $a_m, b_m \in \mathrm{im}(\phi_\gamma)$ and $a_m b_m - b_m a_m \in \mathrm{im}(\phi_{\hat{\gamma}})$. Further we have, that $\mathrm{im}(\phi_{\hat{\gamma}})$ is closed abelian subalgebra by Assertion 5. Hence we have that $a_m b_m - b_m a_m = [a_m, b_m] = 0 \to 0 = [a, b]$. Hence the inductive limit is abelian. $\square$

We now show the analoga of theorem 11 and corollar 3:

Theorem 17 (cp. Ex. 11.5.26 of [18])
Let $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ be an inductive family of C^*-algebras and $(\lim_\to \mathfrak{U}_\gamma, (\phi_\gamma)_{\gamma\in L})$ be its inductive limit. Let $\mathfrak{P}$ be a C^*-algebra together with maps $\tilde{\phi}_\gamma : \mathfrak{U}_\gamma \to \mathfrak{P}$ s.th.

1. *$\forall \gamma \in L : \tilde{\phi}_\gamma$ is an isometric $*$-morphism*
2. *$\forall \gamma' \geq \gamma \in L : \tilde{\phi}_{\gamma'} \circ \phi_{\gamma'\gamma} = \tilde{\phi}_\gamma$.*
3. *$\bigcup_{\gamma\in L} \mathrm{im}(\tilde{\phi}_\gamma)$ is dense in $\mathfrak{P}$.*

Then the universal map $u : \lim_\to \mathfrak{U}_\gamma \to \mathfrak{P}$ is an isometric $*$-isomorphism. Hence $\lim_\to \mathfrak{U}$ and $\mathfrak{P}$ are isomorphic. If further $\mathfrak{U}_\gamma, \phi_{\gamma'\gamma}, \mathfrak{P}, \tilde{\phi}_\gamma$ are unital, then u is also unital.

Proof. The proof of theorem 11 gives, that u is a bijective isometry. We hence have to show, that it is a $*$-morphism. Since the universal map was shown to be an $*$-morphism, the claim follows with lemma 6 directly. In the unital case we have also shown, that u is unital. Further the map extends with lemma 6 to a unital map. Hence the claim follows also in the unital case. $\square$

We now want to investigate, as in the case of Banach spaces, an easy example, which is illustrative by its own right, but also is needed in further applications:

Lemma 11 (cp. [35])
Let $\mathfrak{U}$ be a C^*-algebra, let L be a label set and let $(\mathfrak{U}_\gamma)_{\gamma in L}$ be a family of C^*-algebras with:

1. *$\forall \gamma \in L : \mathfrak{U}_\gamma \subset \mathfrak{U}$ is a subalgebra.*
2. *$\forall \gamma \leq \gamma' \in L : \mathfrak{U}_\gamma \subset \mathfrak{U}_{\gamma'}$*

Then:

1. *$(\mathfrak{U}_\gamma, i_{\gamma\gamma'})_{\gamma',\gamma\in L}$, with $i_{\gamma\gamma'} : \mathfrak{U}_\gamma \hookrightarrow \mathfrak{U}_{\gamma'}$ being the canonical injection, is an inductive system in the category of C^*-algebras.*
2. *The inductive limit of this inductive family is given by:*
$$\lim_{\to} \mathfrak{U}_\gamma = \overline{\bigcup_{\gamma\in L} \mathfrak{U}_\gamma} \subset \mathfrak{U}$$
together with the obvious maps $i_\gamma : \mathfrak{U}_\gamma \to \lim_\to \mathfrak{U}_\gamma$.
3. *1.) and 2.) hold also in $\underline{Cst_{\mathbb{1}}}$.*

Proof. 1.) Since the canonical injection is induced by the embedding of a subalgebra, it is a $*$-morphism.
2.) We first show, that $\bigcup_{\gamma\in L} \mathfrak{U}_\gamma \subset \mathfrak{U}_\gamma$ is a pre C^*-algebra. Therefore observe first, that $\bigcup_{\gamma\in L} \mathfrak{U}_\gamma$ is closed under the algebraic operations: Let $\gamma, \gamma' \in L$ and $a_\gamma \in \mathfrak{U}_\gamma, b_\gamma \in \mathfrak{U}_{\gamma'}$.

Then there is a $\hat{\gamma} \geq \gamma, \gamma'$ such that $a_{\hat{\gamma}} = i_{\hat{\gamma}\gamma} a_\gamma = a_\gamma \in \mathfrak{U}_{\hat{\gamma}}$ and the analogous statement is true for $b_{\gamma'}$. Then we have:

$$a_\gamma b_{\gamma'} = a_{\hat{\gamma}} b_{\hat{\gamma}} \in \mathfrak{U}_{\hat{\gamma}}$$
$$\alpha a_\gamma + \beta b_{\gamma'} = \alpha a_{\hat{\gamma}} + \beta b_{\hat{\gamma}} \in \mathfrak{U}_{\hat{\gamma}}$$

Hence $\bigcup_{\gamma \in L} \mathfrak{U}_\gamma$ is algebraically closed. Since the norm is inherited from $\mathfrak{U}$ it is automatically a pre C^*-algebra. Hence $\overline{\bigcup_{\gamma \in L} \mathfrak{U}_\gamma}$ is a C^*-algebra. Further the canonical injection is automatically an $*$-morphism, since it is induced by the inclusion of a subalgebra. Since automatically $i_{\gamma'} \circ i_{\gamma'\gamma} = i_\gamma$ holds and $\bigcup_{\gamma \in L} \mathfrak{U}_\gamma$ is by definition dense in $\overline{\bigcup_{\gamma \in L} \mathfrak{U}_\gamma}$, the claim follows with theorem 17.
3.) $\bigcup_{\gamma \in L} \mathfrak{U}_\gamma$ is unital, since it is an union of unital subalgebras and since the unit element is unique on $\mathfrak{U}$. By the uniqueness of the unit element, it also follows, that the i_γ and $i_{\gamma'\gamma}$ are unital $*$-morphisms. $\square$

Now we want to investigate, how the spectrum of the inductive limit of unital, commutative C^*-algebras is related to the projective limit of the corresponding spectra. Therefore we state first a well-known theorem from general topology:

Lemma 12 (Urysohn's Lemma, cp. [13])
Let X be a compact Hausdorff space and $X_1, X_2 \subset X$ be disjoint and closed. Then there is a continuous map $f : X \to \mathbb{R}$, such that

$$\forall x \in X_1 : f(x) = 0$$
$$\forall x \in X_2 : f(x) = 1$$
$$\forall x \in X \setminus (X_1 \cup X_2) : f(x) \in [0, 1]$$

We now show, that an injective $*$-morphism between unital, commutative C^*-algebras induces a dual continuous surjection between the spectra:

Lemma 13
Let $\mathfrak{P}$, $\mathfrak{U}$ be unital, commutative C^*-algebras and $i : \mathfrak{P} \hookrightarrow \mathfrak{U}$ be an injective, unital $*$-morphism. Then: The map $i^* : \Delta(\mathfrak{U}) \to \Delta(\mathfrak{P})$ defined as

$$i^* : \Delta(\mathfrak{U}) \to \Delta(\mathfrak{P}), \chi_{\mathfrak{U}} \mapsto \chi_{\mathfrak{U}} \circ i$$

is surjective.

Proof. First observe, that $\mathrm{im}(i^*) \subset \Delta(\mathfrak{P})$ follows, since $\chi_{\mathfrak{U}} \circ i(\mathbb{1}_{\mathfrak{P}}) \neq 0$ for $\forall \chi_{\mathfrak{U}} \in \Delta(\mathfrak{U})$ follows by unitality of $\mathfrak{U}, \mathfrak{P}$ and i. We further show, that i^* is continuous in Gel'fand topology. Therefore let $(\chi_n)_{n \in I} \subset \Delta(\mathfrak{U})$ be a net with $\lim \chi_n = \chi \in \Delta(\mathfrak{U})$. This means:

$$\forall a \in \mathfrak{U} : \forall \epsilon > 0 \exists N \in I : \forall n \geq N : |\chi_n(a) - \chi(a)| < \epsilon \tag{3.3}$$

Now we have, that $i(\mathfrak{P}) \subset \mathfrak{U}$ and further by injnectivity of I for all $a \in i(\mathfrak{P})$ there is exactly one $b \in \mathfrak{P}$ with $i(b) = a$. Hence eq. (3.3) implies:

$$\forall b \in \mathfrak{P} : \forall \epsilon > 0 : \exists N \in I : \forall n \geq N : |\chi_n(i(b)) - \chi(i(b))| < \epsilon$$

Hence we have $\lim i^* \chi_n = \chi$ and hence i^* is continuous.
We now know, that $\Delta(\mathfrak{U})$ and $\Delta(\mathfrak{P})$ are compact as the spectra of unital C^*-algebras. Hence $i^*(\Delta(\mathfrak{U}))$ is closed, since i^* is continuous. Hence we can find a function f, which

vanishes on $i^*(\Delta(\mathfrak{U}))$ but does not vanish on all $\Delta(\mathfrak{P})$ by lemma 12. We have, that the inverse Gel'fand transform of this function is a nonzero element of$\mathfrak{P}$, since f does not vanish on all $\Delta(\mathfrak{P})$.

But now we have that $f(i^*\chi) = 0$ holds for all $\chi \in \Delta(\mathfrak{U})$, by lemma 12. This gives, that for all $\chi \in \Delta(\mathfrak{U})$ $i^*\chi(\check{f})$ must hold. Now we have:

$$i^*\chi(\check{f}) = \chi\left(i(\check{f})\right)$$

Since the spectrum of $\mathfrak{U}$ is the set of all non-zero $*$-morphisms, we know:

$$\forall \chi \in \Delta(\mathfrak{U}) : \chi(i(\check{f})) = 0 \Rightarrow i(\check{f}) = 0$$

Since i is injective, we know, that $i(\check{f}) = 0 \Rightarrow \check{f} = 0$, which is a contradiction. □

We now use this to show the central theorem of this section:

Theorem 18 (cp. [30])
Let $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})$ be an inductive family in $\underline{Cst}_{\mathbb{1}}$. Then: $\left(\Delta(\mathfrak{U}_\gamma), p_{\gamma\gamma'}\right)_{\gamma,\gamma' \in L}$ with $\forall \gamma' \geq \gamma$: $p_{\gamma\gamma'} : \Delta(\mathfrak{U}'_\gamma) \to \Delta(\mathfrak{U}_\gamma), \chi_{\gamma'} \mapsto \chi_{\gamma'} \circ \phi_{\gamma'\gamma}$ is a projective family in the category of compact Hausdorff spaces and further

$$\Delta\left(\lim_{\rightarrow} \mathfrak{U}_\gamma\right) = \lim_{\leftarrow} \Delta(\mathfrak{U}_\gamma)$$

holds.

Proof. We first show, that $\left(\Delta(\mathfrak{U}_\gamma), p_{\gamma\gamma'}\right)_{\gamma,\gamma' \in L}$ defines a projective family. Therefore we have to show, that $p_{\gamma\gamma'}$ as defined above is surjective, continuous in Gel'fand topology and fullfills the composition properties demanded in the definition of a projective family.

That $p_{\gamma\gamma'}$ is surjective and continuous follows directly with the last lemma, since $p_{\gamma\gamma'} = \phi^*_{\gamma'\gamma}$ and $\phi_{\gamma'\gamma}$ is injective and unital.

Further observe: $p_{\gamma\gamma}(\chi) = \chi \circ \phi_{\gamma\gamma} = \chi$ and $p_{\gamma\gamma'} \circ p_{\gamma'\gamma''}(\chi) = \chi \circ \phi_{\gamma''\gamma'} \circ \phi_{\gamma'\gamma} = \chi \circ \phi_{\gamma''\gamma} = p_{\gamma\gamma''}(\chi)$, and hence it satisfies the composition properties of a projective family.

We now want to show, that $\Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma) = \lim_{\leftarrow} \Delta(\mathfrak{U}_\gamma)$. We therefore construct first the conjectured homeomorphism Ψ.

Let $\bar{\chi} \in \lim_{\leftarrow} \Delta(\mathfrak{U}_\gamma)$. Then define $*$-morphism $\chi : \bigcup_\gamma \operatorname{im}(\phi_\gamma) \to \mathbb{C}$ via $\chi = (p_\gamma \bar{\chi})\phi_\gamma^{-1}(\cdot)$. Now we have by lemma 9, that there is an unique extension $\tilde{\chi}$ of χ defined on all $\lim_{\rightarrow} \mathfrak{U}_\gamma$. We then map $\Psi : \bar{\chi} \mapsto \tilde{\chi}$. That its definition is independent of the choice of γ follows from the usual discussion using the property, that the label set is directed. We now show, that the Ψ is bijective and that its inverse Ψ^{-1} is continuous. Since $\lim_{\leftarrow} \Delta(\mathfrak{U}_\gamma)$ and $\Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma)$ are both compact Hausdorff spaces, it then follows, that the mapping is a homeomorphism.

We first show injectivity. Let $\bar{\chi}_1 \neq \bar{\chi}_2$. Recall, that the projective limit is given by

$$\lim_{\leftarrow} \Delta(\mathfrak{U}_\gamma) = \left\{ (x_\gamma)_{\gamma \in L} \in \prod_{\gamma \in L} \Delta(\mathfrak{U}_\gamma) \middle| \forall \gamma' \geq \gamma : p_{\gamma\gamma'}\chi_{\gamma'} = \chi_\gamma \right\}$$
$$p_\gamma : \lim_{\leftarrow} \Delta(\mathfrak{U}_\gamma) \to \Delta(\mathfrak{U}_\gamma), (x_\gamma)_{\gamma \in L} \mapsto x_\gamma$$

Hence $\bar{\chi}_1 \neq \bar{\chi}_2$ means, that there is a $\gamma \in L$ with $p_\gamma \bar{\chi}_1 \neq p_\gamma \bar{\chi}_2$. Hence we have for $a \in \mathrm{im}(\phi_\gamma) \subset \bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma)$ that $(p_\gamma \bar{\chi}_1)\phi_\gamma^{-1}(a) \neq (p_\gamma \bar{\chi}_2)\phi_\gamma^{-1}(a)$, and hence $\Psi(\bar{\chi}_1) \neq \Psi(\bar{\chi}_2)$. Hence the map is injective.

Now we show surjectivity. Let $\tilde{\chi} \in \Delta(\lim_\rightarrow \mathfrak{U}_\gamma)$. We then obtain a family $(\chi_\gamma)_{\gamma \in L}$ via

$$\chi_\gamma = \tilde{\chi} \circ \phi_\gamma$$

We now have for this family, that $\chi_\gamma \in \Delta(\mathfrak{U}_\gamma)$, since $\tilde{\chi}$ is a non-zero $*$-homomorphism and ϕ_γ is an unital $*$-morphism. Further $p_{\gamma\gamma'}\chi_{\gamma'} = \chi_\gamma$ for all $\gamma' \geq \gamma$ follows , since $p_{\gamma\gamma'} = \phi^*_{\gamma'\gamma}$ and since $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ is an inductive system. Hence we have, that $\bar{\chi} = (\chi_\gamma)_{\gamma \in L} \in \lim_\leftarrow \Delta(\mathfrak{U}_\gamma)$ and $\Psi(\bar{\chi}) = \tilde{\chi}$. Hence the map is surjective.

We now show continuity. Therefore observe, that we have constructed an inverse in the proof of surjectivity. We have:

$$\Psi : \lim_{\leftarrow} \Delta(\mathfrak{U}_\gamma) \rightarrow \Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma), \bar{\chi} \mapsto \widetilde{(p_\gamma \bar{\chi})\phi_\gamma^{-1}}(\cdot)$$
$$\Psi^{-1} : \Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma) \rightarrow \lim_{\leftarrow} \Delta(\mathfrak{U}_\gamma), \tilde{\chi} \mapsto (\tilde{\chi} \circ \phi_\gamma)_{\gamma \in L}$$

where the big tilde denotes the unique extension given by lemma 9. We now show, that for all $\gamma \in L$ it holds, that $p_\gamma \circ \Psi^{-1}$ is continuous. Then the claim follows with lemma 7.

Therefore let $(\tilde{\chi}_n)_{n \in I}$ be a net in $\Delta(\lim_\rightarrow \mathfrak{U}_\gamma)$ with $\lim \tilde{\chi}_n = \tilde{\chi} \in \Delta(\lim_\rightarrow \mathfrak{U}_\gamma)$, i.e.:

$$\forall a \in \lim_{\rightarrow} \mathfrak{U}_\gamma : \forall \epsilon > 0 \exists N \in I : \forall n \geq N : |\tilde{\chi}_n(a) - \tilde{\chi}(a)| < \epsilon \tag{3.4}$$

Now observe fist, that $p_\gamma \circ \Psi^{-1}(\tilde{\chi}_n) = \tilde{\chi}_n \circ \phi_\gamma$. Since $\forall a_\gamma \in \mathfrak{U}_\gamma : \phi_\gamma(a_\gamma) \in \lim_\rightarrow \mathfrak{U}_\gamma$, we have that eq. (3.4) implies directly

$$\begin{aligned} \forall a_\gamma \in \mathfrak{U}_\gamma : \forall \epsilon > 0 \exists N \in I : \forall n \geq N :& |(p_\gamma \circ \Psi^{-1}\tilde{\chi}_n)(a_\gamma) - (p_\gamma \circ \Psi^{-1}\tilde{\chi})(a_\gamma) \\ =& |\tilde{\chi}_n \circ \phi_\gamma(a_\gamma) - \tilde{\chi} \circ \phi_\gamma(a_\gamma)| < \epsilon \end{aligned}$$

and hence $p_\gamma \circ \Psi^{-1}$ is continuous. Hence Ψ^{-1} is bijective and continuous and hence a homeomorphism. □

3.4. Inductive Limits of C^*-Algebras II: The Case of the C^*-Algebra of Continuous Functions

In this section we want to investigate the behaviour of the algebra of continuous functions on compact Hausdorff spaces under inductive limits. On the one hand this example can be understood as an important illustration of the concepts of the last section. On the other hand, it is something like the defining example, since any abelian, unital C^*-algebra is isomorphic to the algebra of continuous functions over some compact Hausdorff space.

But first we state the famous theorem of Stone and Weierstrass, since this is used for the proofs in this section.

Theorem 19 (Stone-Weierstrass, cp. Thm. 5.7 of [28])
Let X be a compact Hausdorff space and $C(X)$ be the C^*-algebra of complex valued functions on X together with the sup norm defined in example 2. Let further $\mathfrak{C} \subset C(X)$ be a subalgebra with:

1. *$\mathfrak{C}$ is a $*$-subalgebra of $C(X)$.*

2. *$\forall f \in \mathfrak{C} : f^* \in \mathfrak{U}$.*

3. *$\mathfrak{C}$ seperates points on X, i.e. $\forall x, y \in \mathfrak{C} : x \neq y : \exists f \in \mathfrak{U} : f(x) \neq f(y)$.*

4. *$\forall x \in X \exists f \in \mathfrak{C} : f(x) \neq 0$.*

Then: $\mathfrak{C}$ is dense in $C(X)$.

With this we show:

Theorem 20
Let $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ be a projective family of compact Hausdorff spaces with projective limit $(\lim_{\leftarrow} X_\gamma, p_\gamma)$. Then:

1. *$(C(X_\gamma), \phi_{\gamma'\gamma} = p^*_{\gamma\gamma'})_{\gamma \leq \gamma' \in L}$ is an inductive family of abelian, unital C^*-algebras, where:*
$$p^*_{\gamma\gamma'} : C(X_\gamma) \to C(X_{\gamma'}), f_\gamma \mapsto f_\gamma \circ p_{\gamma\gamma'}$$

2. *The inductive limit of this inductive family is given by*
$$\lim_{\rightarrow} C(X_\gamma) = C(\lim_{\leftarrow} X_\gamma)$$
together with the maps
$$\phi_\gamma := p^*_\gamma : C(X_\gamma) \mapsto C(\lim_{\leftarrow} X_\gamma), f_\gamma \mapsto f_\gamma \circ p_\gamma$$

Proof. 1.) That each $C(X_\gamma)$ is an abelian, unital C^*-algebra for compact Hausdorff spaces X_γ was already stated in example 2. That $\phi_{\gamma'\gamma}$ is well-defined is also clear, since $f_\gamma \circ p_{\gamma\gamma'}$ is continuous as a composition of continuous functions.

That it is a $*$-morphism follows also easy:

$$\begin{aligned}
&\forall f_\gamma, g_\gamma \in C(X_\gamma) : (f_\gamma g_\gamma) \circ p_{\gamma\gamma'} = (f_\gamma \circ p_{\gamma\gamma'})(g_\gamma \circ p_{\gamma\gamma'}) \\
&\forall f_\gamma, g_\gamma \in C(X_\gamma), \forall \alpha, \beta \in \mathbb{C} : (\alpha f_\gamma + \beta g_\gamma) \circ p_{\gamma\gamma'} = \alpha(f_\gamma \circ p_{\gamma\gamma'}) + \beta(g_\gamma \circ p_{\gamma\gamma'}) \\
&\forall f_\gamma \in C(X_\gamma) : (f^*_\gamma \circ p_{\gamma\gamma'}) = (f_\gamma \circ p_{\gamma\gamma'})^*
\end{aligned}$$

That $\phi_{\gamma'\gamma}$ is isometric follows directly, since $p_{\gamma\gamma'}$ is surjective:

$$\begin{aligned}
\|f_\gamma \circ p_{\gamma\gamma'}\|_{\gamma'} &= \sup_{x_{\gamma'} \in X_{\gamma'}} |(f_\gamma \circ p_{\gamma\gamma'})(x_{\gamma'})| \\
&= \sup_{x_\gamma \in p_{\gamma\gamma'}(X_{\gamma'}) = X_\gamma} |f_\gamma(x_\gamma)| \\
&= \|f_\gamma\|_\gamma
\end{aligned}$$

That $\phi_{\gamma'\gamma}$ is unital follows also directly: Let $\mathbb{1}_\gamma : X_\gamma \mapsto \mathbb{C}, x_\gamma \mapsto 1$ be the unique unit element of $C(X_\gamma)$. We then have $\mathbb{1}_\gamma \circ p_{\gamma\gamma'} : X_{\gamma'} \mapsto \mathbb{C}, x_{\gamma'} \mapsto 1$ and hence $\phi_{\gamma'\gamma}\mathbb{1}_\gamma = \mathbb{1}_{\gamma'}$. Further the composition property demanded in the definition of an inductive family follows directly from the composition property of $p_{\gamma\gamma'}$:

$$\phi_{\gamma''\gamma'} \circ \phi_{\gamma'\gamma} = f \circ p_{\gamma\gamma'} \circ p_{\gamma'\gamma''} = f \circ p_{\gamma\gamma''} = \phi_{\gamma''\gamma} f$$

Hence the assertion is shown.
(2) We want to use theorem 17. Therefore we first have to show, that for all $\gamma \in L$ it holds, that ϕ_γ is an isometric, unital $*$-morphism. Further we have to show, that for all $\gamma' \geq \gamma \in L$ $\phi_{\gamma'} \circ \phi_{\gamma'\gamma} = \phi_\gamma$ holds and finally, that $\bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma)$ is dense in $C(\lim_{\leftarrow} X_\gamma)$. That ϕ_γ is an isometric unital $*$-morphism and satisfies the composition properties follows by the same argumentation as in 1.). It remains to show, that $\bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma)$ is dense in $C(\lim_{\leftarrow} X_\gamma)$. Therefore we use the theorem 19. I.e. we want to show, that the algebra

$$\mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma) := \bigcup_{\gamma \in L} p_\gamma^* C(X_\gamma) \subseteq C(\lim_{\leftarrow} X_\gamma)$$

satisfies the prerequisits of theorem 19 in $C(\lim_{\leftarrow} X_\gamma)$.

Therefore we show first, that it is point-separating on $\lim_{\leftarrow} X_\gamma$. Therefore let $(x_\gamma)_{\gamma \in L} \neq (y_\gamma)_{\gamma \in L} \in \lim_{\leftarrow} X_\gamma$. Now this means, that there is a $\gamma \in L$ with $x_\gamma \neq y_\gamma$. We now have, that $C(X_\gamma)$ is point-separating on X_γ, i.e. $\exists f_\gamma \in C(X_\gamma) : f_\gamma(x_\gamma) \neq f_\gamma(y_\gamma)$. Now we have $\phi_\gamma f_\gamma = f_\gamma \circ p_\gamma \in \mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma)$ and further:

$$\begin{aligned} \phi_\gamma f_\gamma((x_\gamma)_{\gamma \in L}) = f_\gamma \circ p_\gamma((x_\gamma)_{\gamma \in L}) = f_\gamma(x_\gamma) \neq \\ \phi_\gamma f_\gamma((y_\gamma)_{\gamma \in L}) = f_\gamma \circ p_\gamma((y_\gamma)_{\gamma \in L}) = f_\gamma(y_\gamma) \end{aligned}$$

Hence $\mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma)$ separates points on $C(\lim_{\leftarrow X_\gamma})$.

Now we show, that $\forall f \in \mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma) : f^* \in \mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma)$. Therefore let $f \in \mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma)$. Hence there is a $\gamma \in L$ and a $f_\gamma \in C(X_\gamma)$ such that $f = f_\gamma \circ p_\gamma$. Now we have $f_\gamma^* \in C(X_\gamma)$ and $f^* = f_\gamma^* \circ p_\gamma$. Hence $f^* \in \mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma)$.

Finally we want to show, that $\forall x \in \lim_{\leftarrow} X_\gamma : \exists f \in \mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma) : f(x) \neq 0$. Therefore let $(x_\gamma)_{\gamma \in L} \in \lim_{\leftarrow} X_\gamma$. Now choose any $\gamma \in L$. Define the function $\mathbb{1}_\gamma : X_\gamma \to \{1\}, x \mapsto 1$ which exists on compact Hausdorff spaces. Then $\mathbb{1}(x_\gamma) \neq 0$ and hence $\phi_\gamma \mathbb{1}_\gamma((x_\gamma)_{\gamma \in L}) \neq 0$.

Finally we want to show, that $\mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma)$ is a subalgebra. Therefore we have to show, that it is closed under the $*$-algebra structure. But this follows directly since for each $\gamma \in L$ it holds, that $C(X_\gamma)$ is a C^*-Algebra. □

In the literature one often encounters the concept of so called cylindrical functions. We want to introduce this concept here as well and want to understand its connection to the concepts presented in this thesis:

Definition 17 (Cylindrical function, cp. [31], [6])
Let $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ be a projective family of compact Hausdorff spaces and let $(\lim_{\leftarrow} X_\gamma, p_\gamma)$ be its inductive limit. Then define the set of Cylindrical functions corresponding to this family as:

$$\mathrm{Cyl}(\lim_{\leftarrow} X_\gamma) := \bigcup_{\gamma \in L} C(X_\gamma) \Big/ \sim$$

with

$$f_{\gamma_1} \sim f_{\gamma_2} \Leftrightarrow \forall \hat{\gamma} \geq \gamma_1, \gamma_2 : p_{\gamma_1\hat{\gamma}}^* f_{\gamma_1} = p_{\gamma_2\hat{\gamma}}^* f_{\gamma_2}$$

for $\gamma_1, \gamma_2 \in L$ and $f_{\gamma_1} \in C(X_{\gamma_1})$, $f_{\gamma_2} \in C(X_{\gamma_2})$. We further define according to the construction in the last theorem:

$$\mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma) := \bigcup_{\gamma \in L} \mathrm{im}(p_\gamma^*) \subset C(\lim_{\leftarrow} X_\gamma)$$

Please observe that we have shown in the proof of theorem 20:

Corollar 5
Let $(X_\gamma, p_{\gamma\gamma'})_{\gamma\leq\gamma'\in L}$ be a projective family of compact Hausdorff spaces and let $(\lim_\leftarrow X_\gamma, p_\gamma)$ be its inductive limit. Then: $\mathrm{Cyl}'(\lim_\leftarrow X_\gamma)$ *is dense in* $\lim_\rightarrow C(X_\gamma) = C(\lim_\rightarrow X_\gamma)$.

We now have the following lemma:

Lemma 14
Let $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ be a projective family of compact Hausdorff spaces and let $(\lim_\leftarrow X_\gamma, (p_\gamma)_{\gamma\in L})$ be its inductive limit. Then:

1. $\mathrm{Cyl}(\lim_\leftarrow X_\gamma)$ *is an unital $*$-algebra and the canonical projection $\pi : \bigcup_{\gamma\in L} C(X_\gamma) \to$* $\mathrm{Cyl}(\lim_\leftarrow X_\gamma)$ *is an isometric $*$-morphism.*
2. $\mathrm{Cyl}'(\lim_\leftarrow X_\gamma)$ *is an unital $*$-algebra.*
3. $\mathrm{Cyl}(\lim_\leftarrow X_\gamma)$ *and* $\mathrm{Cyl}'(\lim_\leftarrow X_\gamma)$ *are isometrically isomorphic as $*$-algebras.*
4. $\mathrm{Cyl}(\lim_\leftarrow X_\gamma)$ *and* $\mathrm{Cyl}'(\lim_\leftarrow X_\gamma)$ *are dense in $C(X_\gamma)$.*

Proof. 1.) We first define a $*$-algebra structure on $\mathrm{Cyl}(\lim_\leftarrow X_\gamma)$: Let $\gamma_1, \gamma_2 \in L$ and $\hat\gamma \geq \gamma_1, \gamma_2$. Let further $\alpha, \beta \in \mathbb{C}$ and $f_{\gamma_1} \in C(X_{\gamma_1})$ and $g_{\gamma_2} \in C(X_{\gamma_2})$. Let further $\|\cdot\|_\gamma$ denote the sup norm on $C(X_\gamma)$. Then define:

$$\|[f_{\gamma_1}]\| := \|f_{\gamma_1}\|_{\gamma_1} \tag{3.5}$$
$$[f_{\gamma_1}]\cdot[g_{\gamma_2}] := [(p^*_{\gamma_1\hat\gamma} f_{\gamma_1})\cdot(p^*_{\gamma_2\hat\gamma} g_{\gamma_2})] \tag{3.6}$$
$$\alpha[f_{\gamma_1}] + \beta[g_{\gamma_2}] := [\alpha p^*_{\gamma_1\hat\gamma} f_{\gamma_1} + \beta p^*_{\gamma_2\hat\gamma} g_{\gamma_2}] \tag{3.7}$$
$$[f_{\gamma_1}]^* := [f^*_{\gamma_1}] \tag{3.8}$$

We now have to show, that all this is well-defined.

We first show that eq. (3.5) is well defined. Therefore let $f_{\gamma_1} \in C(X_{\gamma_1}), g_{\gamma_2} \in C(X_{\gamma_2})$ and $f_{\gamma_1}, g_{\gamma_2} \in [f_{\gamma_1}]$. Then for $\hat\gamma \geq \gamma_1, \gamma_2$ it holds, that:

$$p^*_{\gamma_1\hat\gamma} f_{\gamma_1} = p^*_{\gamma_2\hat\gamma} g_{\gamma_2}$$

Hence especially:

$$\|p^*_{\gamma_1\hat\gamma} f_{\gamma_1}\| = \|p^*_{\gamma_2\hat\gamma} g_{\gamma_2}\|$$

And since $\forall\gamma' \geq \gamma : p^*_{\gamma\gamma'} : C(X_\gamma) \to C(X_{\gamma'})$ is an isometry, as it was shown in the proof of theorem 20, $\|\cdot\|$ defines a norm on $\mathrm{Cyl}(\lim_\leftarrow X_\gamma)$.
We now only show, that eq. (3.6) is well defined. Then well definedness of eq. (3.7) and eq. (3.8) follow by the same argumentation. We first show, that it is independet of the choice of a upper bound $\hat\gamma$. Therefore let $\hat\alpha \geq \gamma_1, \gamma_2$ and $\hat\alpha \neq \hat\gamma$. Now let $\hat\beta \geq \hat\alpha, \hat\gamma$. Then:

$$p^*_{\hat\gamma\hat\beta}\left((p^*_{\gamma_1\hat\gamma} f_{\gamma_1})(p^*_{\gamma_2\hat\gamma} g_{\gamma_2})\right) = (p^*_{\gamma_1\hat\beta} f_{\gamma_1})(p^*_{\gamma_2\hat\beta} g_{\gamma_2}) =$$
$$p^*_{\hat\alpha\hat\beta}\left((p^*_{\gamma_1\hat\alpha} f_{\gamma_1})(p^*_{\gamma_2\hat\alpha} g_{\gamma_2})\right) = (p^*_{\gamma_1\hat\beta} f_{\gamma_1})(p^*_{\gamma_2\hat\beta} g_{\gamma_2})$$

And hence $(p^*_{\gamma_1\hat\beta} f_{\gamma_1})(p^*_{\gamma_2\hat\beta} g_{\gamma_2}) \sim (p_{\gamma_1\hat\gamma} f_{\gamma_1})(p^*_{\gamma_2\hat\gamma} g_{\gamma_2})$. We now show, that the definition is independent of the representant. Therefore let $\tilde f_{\tilde\gamma_1} \in [f_{\gamma_1}]$ and $\tilde g_{\tilde\gamma_2} \in [g_{\gamma_2}]$. Then for $\tilde\gamma \geq \tilde\gamma_1, \tilde\gamma_2$:

$$[f_{\gamma_1}][g_{\gamma_2}] = [(p^*_{\tilde\gamma_1\tilde\gamma}\tilde f_{\tilde\gamma_1})(p^*_{\tilde\gamma_2\tilde\gamma}\tilde g_{\tilde\gamma_2})]$$

Now let $\beta \geq \tilde{\gamma}, \hat{\gamma}$. Then:

$$p^*_{\tilde{\gamma}\beta}\left(p_{\tilde{\gamma}_1\tilde{\gamma}}\tilde{f}_{\tilde{\gamma}_1})(p^*_{\tilde{\gamma}_2\tilde{\gamma}}\tilde{g}_{\tilde{\gamma}_2})\right) = (p^*_{\tilde{\gamma}_1\beta}\tilde{f}_{\tilde{\gamma}_1})(p^*_{\tilde{\gamma}_2\beta}\tilde{g}_{\tilde{\gamma}_2})$$
$$p^*_{\hat{\gamma}\beta}\left(p_{\gamma_1\hat{\gamma}}f_{\gamma_1})(p^*_{\gamma_2\hat{\gamma}}g_{\gamma_2})\right) = (p^*_{\gamma_1\beta}f_{\gamma_1})(p^*_{\gamma_2\beta}g_{\gamma_2})$$

Now $\tilde{f}_{\tilde{\gamma}_1} \sim f_{\gamma_1}$ and $\tilde{g}_{\tilde{\gamma}_2} \sim g_{\gamma_2}$. With this we have $p^*_{\gamma_1\beta}f_{\gamma_1} = p^*_{\tilde{\gamma}_1\beta}\tilde{f}_{tilde\gamma_1}$ and hence the multiplication is independent of the representant. As already stated above, well definedness of eq. (3.7) and eq. (3.8) follow analogously.

Finally we show, that $\pi|_{C(X_\gamma)}$ is isometric $*$-morphism for all $\gamma \in L$. Since we have already shown, that for all $\gamma \in L$ $\|[f_\gamma]\| = \|f_\gamma\|_\gamma$ holds, we have, that π is isometric. That it is a $*$-morphism follows directly by the definition of the $*$-structure on $\mathrm{Cyl}(\|im_\leftarrow X_\gamma)$, since e.g. for the multiplication $\pi(f_\gamma g_\gamma) = [f_\gamma g_\gamma] = [f_\gamma][g_\gamma]$ holds.

Finally it can be easily shown, that $[\mathbb{1}_{C(X_\gamma)}]$ defines a unit for $\mathrm{Cyl}(\lim_\leftarrow X_\gamma)$, since for all $\gamma' \in L$ it holds, that $\mathbb{1}_{C(X_{\gamma'})} \in [\mathbb{1}_{C(X_\gamma)}]$, since $p^*_{\gamma\gamma'}$ is unital.

2.) We have already discussed in the proof of theorem 20, that $\mathrm{Cyl}'(\lim_\leftarrow X_\gamma)$ is a $*$-subalgebra of $C(\lim_\leftarrow X_\gamma)$ and hence a $*$-algebra. That it is unital follows, since p^*_γ is unital, as discussed in the proof of theorem 20.

3.) We define a map:

$$\Psi : \mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma) \to \mathrm{Cyl}(\lim_{\leftarrow} X_\gamma), p^*_\gamma(f_\gamma) \mapsto \pi(f_\gamma)$$

We have to show, that the map is a bijective, isometric $*$-morphism.

We first show, that it is well-defined. Therefore let $f_{\gamma_1} \in C(X_{\gamma_1})$ and $f_{\gamma_2} \in C(X_{\gamma_2})$ with $p^*_{\gamma_1}f_{\gamma_1} = p^*_{\gamma_2}f_{\gamma_2}$. Let now $\hat{\gamma} \geq \gamma_1, \gamma_2$. Then: $p^*_{\gamma_1} = p^*_{\hat{\gamma}} \circ p^*_{\gamma_i\hat{\gamma}}$. This gives futher:

$$p^*_{\hat{\gamma}}f_{\hat{\gamma}} = p^*_{\gamma_1}f_{\gamma_1} = p^*_{\gamma_2}f_{\gamma_2} = p^*_{\hat{\gamma}}\tilde{f}_{\hat{\gamma}}$$

Where we have defined $f_{\hat{\gamma}} := p^*_{\gamma_1\hat{\gamma}}f_{\gamma_1}$ and $\tilde{f}_{\hat{\gamma}} := p^*_{\gamma_2\hat{\gamma}}f_{\gamma_2}$. Since p^*_γ is an injective isometry, as shown in the proof of theorem 20, this gives, that $f_{\hat{\gamma}} = \tilde{f}_{\hat{\gamma}}$. Hence we have, that for all $\hat{\gamma} \geq \gamma_1, \gamma_2$ it holds, that $p^*_{\gamma_1\hat{\gamma}}f_{\gamma_1} = p^*_{\gamma_2\hat{\gamma}}$ which gives $\pi(f_{\gamma_1}) = \pi(f_{\gamma_2})$. Hence Ψ is right-unique. It is left-total by definition of $\mathrm{Cyl}'(\lim_\leftarrow X_\gamma)$. Hence Ψ is well-defined.

Surjectivity follows directly by the fact, that π is surjective as a quotient map. Further Ψ is an isometric $*$-morphism, since p^*_γ and π are isometric $*$-morphism. Hence the claim follows.

4.) Since it was proven in the last theorem, that $\mathrm{Cyl}'(\lim_\leftarrow X_\gamma)$ is dense in $C(\lim_\leftarrow X_\gamma)$, this follows with (3.). □

3.5. Inductive Limits of Hilbert Spaces II: Measure Spaces and L^2 Spaces

We first want to define the category of measure spaces as used in this thesis:

Definition 18
Let $\underline{MH}$ be the category whose objects are tuples of the form $(X, \sigma, d\mu)$ (or $(X, d\mu)$ for short), such that

1. *X is a compact Hausdorff space,*
2. *σ is the σ-algebra of Borel sets thereon,*
3. *$d\mu$ is a regular, Borel probability measure on (X, σ)*

and whose morphisms are continuous surjections $p : (X_1, d\mu_1) \to (X_2, d\mu_2)$ for which additionally

$$p_*\mu_1 := \mu_1 \circ p^{-1} = \mu_2$$

holds.

We now show the following theorem:

Theorem 21 (Projective limit exists in the category of measure spaces, cp. [31])
The projective limit exists in $\underline{MH}$ and is unique. In this case the projective limit measure space is denoted by $(\lim_{\leftarrow} X_\gamma, d\lim_{\leftarrow} \mu_\gamma)$.

Proof. Let $\big((X_\gamma, \sigma_\gamma, \mu_\gamma), p_{\gamma\gamma'}\big)_{\gamma,\gamma' \in L}$ be a projective family in $\underline{MH}$. Let further $\lim_{\leftarrow} X_\gamma$ together with maps p_γ be the unique projective limit of the projective family $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ in the category of compact Hausdorff spaces. Let σ be the Borel σ-algebra thereon. We now want to construct a measure μ on $(\lim_{\leftarrow} X_\gamma, \sigma)$ satisfying $p_{\gamma *}\mu = \mu_\gamma$. We therefore define the following functional:

$$\Lambda : \mathrm{Cyl}(\lim_{\leftarrow} X) \to \mathbb{C}, [f_\gamma] \mapsto \int_{X_\gamma} f_\gamma d\mu_\gamma$$

We now show, that this functional is continuous, positive and independent of the choice of a representative. Continuity follows, since $\overline{\mathrm{Cyl}(\lim_{\leftarrow} X)}$ is a unital C^*-algebra, and functionals on subsets of unital C^*-algebras are automatically continuous (cp. p. 221 of [31]). Further also independence of the choice of a representative follows directly by the usual discussion using, that L is a directed, partially ordered set. Positivity follows, since each $d\mu_\gamma$ is a probability measure and the normalization $\Lambda(\mathbb{1}_{\mathrm{Cyl}(\lim_{\leftarrow} X_\gamma)}) = 1$ follows, since each $d\mu_\gamma$ is a probability measure and since the unit on $\mathrm{Cyl}(\lim_{\leftarrow} X_\gamma)$ is given by $[\mathbb{1}_{C(X_\gamma)}$. By lemma 5 Λ can be extended unambiguously to a functional on $\overline{\mathrm{Cyl}(\lim_{\leftarrow} X)}$ which gives by lemma 14 a positive, normalized functional on $C(\lim_{\leftarrow}(X))$. Then claim then follows by theorem 7. □

We finally show the following compatibility theorem:

Theorem 22
Let $\big((X_\gamma, d\mu_\gamma), p_{\gamma\gamma'}\big)_{\gamma' \geq \gamma \in L}$ be a projective family of Measure spaces. Then:

1. *$\Big(L^2(X_\gamma, d\mu_\gamma), \psi_{\gamma'\gamma} := p^*_{\gamma\gamma'}\Big)_{\gamma' \geq \gamma \in L}$ is an inductive family in the category of Hilbert spaces.*
2. *$\lim_{\to} L^2(X_\gamma, d\mu_\gamma) = L^2(\lim_{\leftarrow} X_\gamma, d\lim_{\leftarrow} \mu_\gamma)$.*

Proof. (1) Please observe first, that $\psi_{\gamma'\gamma}$ is defined on a dense subset $C(X_\gamma)$ and further it maps on a dense subset $C(X_{\gamma'})$. We now want to show, that $\psi_{\gamma'\gamma}$ defines an isometry on $C(X_\gamma)$ together with the L^2 norm. Therefore let $f \in C(X_\gamma)$. Then it follows by the change of variables formula:

$$\int_{X_{\gamma'}} \psi_{\gamma'\gamma} f d\mu_{\gamma'} = \int_{X_\gamma} f d\left(p_{\gamma\gamma'*}\mu_{\gamma'}\right) = \int_{X_\gamma} f d\mu_\gamma$$

Now it follows by lemma 5 and by denseness of $C(X_\gamma) \subset L^2(X_\gamma, d\mu_\gamma)$, that $\psi_{\gamma'\gamma}$ defines an isometry $\psi_{\gamma'\gamma} : L^2(X_\gamma, d\mu_\gamma) \to L^2(X_{\gamma'}, d\mu_{\gamma'})$. We now want to show, that $\psi_{\gamma'\gamma}$ satisfies the composition properties demanded in the definition of an inductive family. By uniqueness of the extension given by lemma 5, it suffices to show them on a dense set. But since $\psi_{\gamma'\gamma} := p^*_{\gamma\gamma'}$, the composition properties follow directly, since $p_{\gamma\gamma'}$ satisfies the composition properties demanded in the definition of an projective family.
(2) We use the universal property of the inductive limit together with theorem 11.

Define the following maps:

$$\psi_\gamma : C(X_\gamma) \subset L^2(X_\gamma, d\mu_\gamma) \to L^2(\lim_{\leftarrow} X_\gamma, \lim_{\leftarrow} d\mu_\gamma), f_\gamma \mapsto f_\gamma \circ p_\gamma$$

where $p_\gamma : \lim_{\leftarrow} X_\gamma \to X_\gamma$ is the surjective continuous mapping corresponding to $\lim_{\leftarrow} X_\gamma$. We now have to show, that ψ_γ extends uniquely to $L^2(X_\gamma, d\mu_\gamma)$ and that this extension satisfies the prerequisits of theorem 11. Then the claim follows with theorem 11.

Assertion 1: ψ_γ is a isometry and extends uniquely to $L^2(X_\gamma, d\mu_\gamma)$.

Proof of Assertion 1: We show, that ψ_γ is an isometry on $C(X_\gamma)$ on together with the L^2 norm. Then the assertion follows with lemma 5.
Therefore:

$$\begin{aligned}
\|\psi_\gamma f_\gamma\| &= \int_{\lim_{\leftarrow} X_\gamma} f_\gamma \circ p_\gamma d \lim_{\leftarrow} \mu_\gamma \\
&= \int_{X_\gamma} f_\gamma d(p_{\gamma *} \leftarrow \mu_\gamma) \\
&= \int_{X_\gamma} f_\gamma d\mu_\gamma \\
&= \|f_\gamma\|_\gamma
\end{aligned}$$

Hence we have, that ψ_γ is an isometry and hence extends uniquely to an isometry on $L^2(X_\gamma, d\mu_\gamma)$ by lemma 5.

Assertion 2: The unique bounded extension of ψ_γ satisfies the prerequisits of theorem 11, i.e.

1. $\forall \gamma \in L : \psi_\gamma : L^2(X_\gamma, d\mu_\gamma) \to L^2(\lim_{\leftarrow} X_\gamma, \lim_{\leftarrow} d\mu_\gamma)$ is an isometry.

2. $\forall \gamma' \geq \gamma \in L : \psi_{\gamma'} \circ \psi_{\gamma'\gamma} = \psi_\gamma$.

3. $\bigcup_{\gamma \in L} \operatorname{im}(\psi_\gamma)$ is dense in $L^2(\lim_{\leftarrow} X_\gamma, \lim_{\leftarrow} d\mu_\gamma)$

Proof of Assertion 2: Assertion 2.1 was already shown in Assertion 1.
Assertion 2.2 follows also directly, since $\psi_{\gamma'\gamma} = p^*_{\gamma\gamma'}$, $\psi_\gamma = p^*_\gamma$ and with this:

$$\psi_{\gamma'}\psi_{\gamma'\gamma} = p^*_{\gamma'}p^*_{\gamma\gamma'} = p^*_\gamma = \psi_\gamma$$

We now show Assertion 2.3: We know from lemma 14, that $\mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma) := \bigcup_{\gamma\in L} p^*_\gamma\,(C(X_\gamma))$ is dense in $C(\lim_{\leftarrow} X_\gamma)$ with respect to the sup norm. Further we have, that for all $\gamma \in L$ it holds, that $C(X_\gamma) \subset L^2(X_\gamma, d\mu_\gamma)$. Hence $\mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma) \subset \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$ does hold. We now show the following: Let X be a compact Hausdorff space and $\mathfrak{C} \subset C(X)$ be dense with respect to $\|\cdot\|_\infty$. Then it is dense with respect to $\|\cdot\|_2$. Therefore let $f \in C(X)$ and $(f_n)_{n\in\mathbb{N}} \subset \mathfrak{C}$ with $f_n \to f$ with respect to $\|\cdot\|_\infty$, i.e.

$$\forall\epsilon' > 0 \exists N \in \mathbb{N} \forall n \geq N : \|f_n - f\|_\infty < \epsilon' \tag{3.9}$$

Now we have for all $f \in C(X)$:

$$\begin{aligned}\|f\|_2 &= \left(\int_X |f|^2 d\mu\right)^{\frac{1}{2}} \\ &\leq \left(\int_X d\mu \|f\|^2_\infty\right)^{\frac{1}{2}} \\ &= (Vol(X))^{\frac{1}{2}}\|f\|_\infty\end{aligned}$$

Now let $\epsilon > 0$. Then we have for N and $n \geq N$ as in eq. (3.9) for $\epsilon' = \frac{\epsilon}{\sqrt{Vol(X)}}$, that:

$$\|f_n - f\|_2 \leq \sqrt{Vol(X)}\|f_n - f\|_\infty \leq \epsilon$$

and hence $\mathfrak{C}$ is dense in $C(X)$ w.r.t. $\|\cdot\|_2$.

With this we have now, that $\mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma)$ is dense in $C(\lim_{\leftarrow} X_\gamma)$ with respect to $\|\cdot\|_2$. But since $\mathrm{Cyl}'(\lim_{\leftarrow} X_\gamma) \subset \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$, we also have, that the latter is dense in $C(\lim_{\leftarrow} X_\gamma)$. Now Assertion 2.3 follows, since $C(\lim_{\leftarrow})$ is dense in $L^2(\lim_{\leftarrow} X_\gamma, d\lim_{\leftarrow}\mu_\gamma)$ and denseness is transitive.

Now the claim follows with theorem 11. □

3.6. Inductive Limits of C^*-Algebras III: States, Representations and the Gel'fand transform

In this section we want to analyze the behaviour of states, representations and the Gel'fand transform under inductive limits.

Lemma 15 (BLT for states)
Let $\mathfrak{U}$ be a C^*-algebra and $V \subset \mathfrak{C}$ be a dense $*$-subalgebra. Let $\omega : V \to \mathbb{C}$ be a state. In the case where $\mathfrak{U}$ is unital, we further assume, that $\mathbb{1}_{\mathfrak{U}} \subset V$.
Then: There exists a unique state $\tilde{\omega}$ on $\mathfrak{C}$ which extends ω.

Proof. By lemma 5 the unique extension of ω is a linear functional. Hence it remains to show, that $\tilde{\omega}$ is positive and that $\|\tilde{\omega}\| = 1$. Further we have to show, that for the unital case $\tilde{\omega}(\mathbb{1}_{\mathfrak{U}}) = 1$ holds.

Therefore first observe, that for a sequence $(a_n)_{n\in\mathbb{N}} \subset V$ with $a_n \to a \in \mathfrak{U}$ the inequality $\omega(a_n^* a_n) \geq 0$ holds for all $n \in \mathbb{N}$ and hence $\tilde{\omega}(a^* a) = \lim_{n\to\infty} \omega(a_n^* a_n) \geq 0$.

Further observe, that $\|\tilde{\omega}\| = \sup_{a\in V\setminus\{0\}} \frac{|\omega(a)|}{\|a\|}$ and that the supremum of a continuous function over a dense subset equals the supremum over the full set. Finally in the unital case $\tilde{\omega}(\mathbb{1}_{\mathfrak{U}}) = 1$ follows since $\tilde{\omega}|_V = \omega$ holds. □

We now define the notion of a projective family family of states:

Definition 19
Let $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive family of C^*-algebras with inductive limit $(\lim_\rightarrow \mathfrak{U}_\gamma, (\phi_\gamma)_{\gamma\in L})$. Then:

1. *A projective family of states is a family $(\omega_\gamma)_{\gamma\in L}$ such that:*
   a) *For each $\gamma \in L$ it holds, that ω_γ is a state on $\mathfrak{U}_\gamma$.*
   b) *$\forall \gamma' \geq \gamma : \omega_\gamma = \omega_{\gamma'} \circ \phi_{\gamma'\gamma}$.*
2. *Let $(\omega_\gamma)_{\gamma\in L}$ be a projective family of states. Then its projective limit is a state ω on $\lim_\rightarrow \mathfrak{U}_\gamma$ such that $\omega_\gamma = \omega \circ \phi_\gamma$ holds for all $\gamma \in L$.*

We now investigate existence and uniqueness of projective limits for states:

Theorem 23
Let $(\omega_\gamma)_{\gamma\in L}$ be a projective family of states corresponding to an inductive family of C^*-algebras $\mathfrak{U}_\gamma$. Then its projective limit exists and is unqiue.

Proof. Define first the conjectured projective limit as a linear functional on the dense set $\bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma)$ via

$$\chi : \bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma) \to \mathbb{C}, a \mapsto \chi_\gamma\left(\phi_\gamma^{-1}(a)\right)$$

By the usual discussion this definition is independent of the choice of γ. That χ is a positive linear functional with $\|\chi\| = 1$ follows, since χ_γ is a state and ϕ_γ is an isometric $*$-morphism. That the unique bounded extension exists and defines a state follows with lemma 15. Further we have, that ψ_γ is unital in the unital case and hence $\omega(\mathbb{1}_{\lim_\rightarrow \mathfrak{U}_\gamma}) = 1$ follows in the unital case.

Now we have to show, that this really defines a projective limit. This is indeed the case, since:

$$\chi \circ \phi_\gamma = \chi_\gamma \circ \phi_\gamma^{-1} \circ \phi_\gamma = \chi_\gamma$$

Further we have to show, that the projective limit is unique. Therefore assume there is another state $\bar{\chi} \neq \chi$ on $\lim_\rightarrow \mathfrak{U}_\gamma$ for which $\bar{\chi} \circ \phi_\gamma = \chi_\gamma$ holds. Hence there must be an $a \in \lim_\rightarrow \mathfrak{U}_\gamma$ with $\bar{\chi}(a) \neq \chi(a)$. But now for all $b \in \bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma)$ we have, that $\chi(b) = \chi \circ \phi_\gamma(b_\gamma) = \omega_\gamma(b_\gamma) = \bar{\chi} \circ \phi_\gamma(b_\gamma) = \bar{\chi}(b)$. Hence $\chi = \bar{\chi}$ must hold on $\bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma)$. Since the extension delivered by lemma 15 is unique, this gives a contradiction and hence the projective limit is unique.

Hence the claim follows. □

We now introduce the notion of inductive families in the context of representations:

Definition 20 (cp. Ex. 11.5.28 of [18])
Let $(\mathcal{H}_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ be an inductive family of Hilbert spaces and $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ be an inductive family of C^*-algebras. Let further $(\lim_\rightarrow \mathcal{H}_\gamma, \psi_\gamma)$ and $(\lim_\rightarrow \mathfrak{U}_\gamma, \phi_\gamma)$ denote the corresponding inductive limits. Then:

1. *A family $(\pi_\gamma)_{\gamma \in L}$ is called an inductive family of representations, iff:*
   a) *For each $\gamma \in L$ we have, that π_γ is a representation of $\mathfrak{U}_\gamma$ on $\mathcal{H}_\gamma$.*
   b) $\forall \gamma' \geq \gamma \in L : \forall a_\gamma \in \mathfrak{U}_\gamma : \pi(\phi_{\gamma'\gamma} a_\gamma)\psi_{\gamma'\gamma} = \psi_{\gamma'\gamma}\pi_\gamma(a_\gamma)$
2. *For an inductive family of representations $(\pi_\gamma)_{\gamma \in L}$, the inductive limit of this family is a representation $\lim_\rightarrow \pi_\gamma$ of $\lim_\rightarrow \mathfrak{U}_\gamma$ on $\lim_\rightarrow \mathcal{H}_\gamma$ such that*

$$\forall a_\gamma \in \mathfrak{U}_\gamma : \left[\lim_{\rightarrow} \pi_\gamma\right] (\phi_\gamma a_\gamma)\psi_\gamma = \psi_\gamma \pi_\gamma(a_\gamma)$$

Now we show first the following theorem:

Theorem 24
Let $(\pi_\gamma)_{\gamma \in L}$ be an inductive family of representations corresponding to an inductive family of C^*-algebras $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ and an inductive family of Hilbert spaces $(\mathcal{H}_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$. Then its projective limit exists and is unqiue.

Proof. We construct first for each $\gamma_0 \in L$ a representation of $\mathfrak{U}_{\gamma_0}$ on $\lim_\rightarrow \mathcal{H}_{\gamma_0}$, which we call ρ_{γ_0}:

$$\forall a_{\gamma_0} \in \mathfrak{U}_{\gamma_0} : \rho_{\gamma_0}(a_{\gamma_0}) = \lim_{\rightarrow} \left[\pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0})\right]$$

Here the latter $\lim_\rightarrow$ denotes an inductive limit on an inductive family of operators. We now show the following Assertions:

Assertion 1: $(\pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0}))_{\gamma \geq \gamma_0}$ is for each $\gamma_0 \in L$ an inductive family of operators.

Proof of Assertion 1: Since π_γ is a representation mapping on $\mathcal{B}(\mathcal{H}_\gamma)$ we have, that $\forall \gamma \geq \gamma_0 : \pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0}) \in \mathcal{B}(\mathcal{H}_\gamma)$. Further we have, that representations are norm contracting (cp. corollar 1) and hence:

$$\sup_{\gamma \in L} \|\pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0})\| \leq \sup_{\gamma \in L} \|\phi_{\gamma\gamma_0} a_{\gamma_0}\| = \|a_{\gamma_0}\| < \infty$$

Finally we have:

$$\pi_{\gamma'}(\phi_{\gamma'\gamma_0} a_{\gamma_0})\psi_{\gamma'\gamma} = \pi_{\gamma'}(\phi_{\gamma'\gamma}\phi_{\gamma\gamma_0} a_{\gamma_0})\psi_{\gamma'\gamma} = \psi_{\gamma'\gamma}\pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0})$$

and hence it is an inductive family of operators.

Assertion 2: Let ρ be a representation of a C^*-algebra $\mathfrak{U}$ on a dense subset $V \subset \mathcal{H}$ of a Hilbert space $\mathcal{H}$. Then it extends uniquely to a representation on all $\mathcal{H}$. (Basically we show, that the claim holds in the more general situation, where ρ is a representation of a $*$-algebra $\mathfrak{U}$ on a dense subset $V \subset \mathcal{H}$ of a Hilbertspace $\mathcal{H}$ and $\rho(\mathfrak{U}) \subset \mathcal{B}(\mathcal{H})$).

Proof of Assertion 2: By the lemma 5 we have, that for all $a \in \mathfrak{U}$ the unique extension of $\pi(a)$ for $f \in \mathcal{H}$ is given by

$$\lim_{n\to\infty} \pi(a)f_n =: \pi(a)f$$

for a sequence $(f_n)_{n\in\mathbb{N}} \subset V$ with $f_n \to f(n \to \infty)$. Since the representation is defined on a dense subset we have:

$$\begin{aligned}\pi(\alpha a + \beta b)f &= \lim_{n\to\infty} \pi(\alpha a + \beta b)f_n \\ &= \alpha \lim_{n\to\infty} \pi(a)f_n + \beta \lim_{n\to\infty} \pi(b)f_n \\ &= \alpha\pi(a)f + \beta\pi(b)f\end{aligned}$$

Further

$$\begin{aligned}\pi(ab)f &= \lim_{n\to\infty} \pi(ab)f_n \\ &= \lim_{n\to\infty} \pi(a)\pi(b)f_n \\ &= \pi(a)\pi(b)f\end{aligned}$$

and:

$$\begin{aligned}\pi(a^*)f &= \lim_{n\to\infty} \pi(a^*)f_n \\ &= \lim_{n\to\infty} \pi(a)^*f_n \\ &= \pi(a)^*f\end{aligned}$$

Hence the claim follows.

Assertion 3: The map $\rho_{\gamma_0} : \mathfrak{U}_{\gamma_0} \to \mathcal{B}(\lim_\to \mathcal{H}_\gamma), a_{\gamma_0} \mapsto \lim_\to[\pi_\gamma(\phi_{\gamma\gamma_0}a_{\gamma_0})]$ defines a representation of $\mathfrak{U}_{\gamma_0}$ on $\lim_\to \mathcal{H}_\gamma$.

Proof of Assertion 3: We show, that it defines a representation on $\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$. Then the claim follows by the last assertion. We therefore have to show, that $\forall f \in \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$ the following holds:

1. $\forall a_{\gamma_0} \in \mathfrak{U}_{\gamma_0} : \rho_{\gamma_0}(a_{\gamma_0}) \in \mathcal{B}\left(\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)\right)$.
2. $\forall \alpha, \beta \in \mathbb{C} : \forall a_{\gamma_0}, b_{\gamma_0} \in \mathfrak{U}_{\gamma_0} : \rho_{\gamma_0}(\alpha a_{\gamma_0} + \beta b_{\gamma_0})f = \alpha\rho_{\gamma_0}(a_{\gamma_0})f + \beta\rho_{\gamma_0}(b_{\gamma_0})f$.
3. $\forall a_{\gamma_0}, b_{\gamma_0} \in \mathfrak{U}_{\gamma_0} : \rho_{\gamma_0}(a_{\gamma_0}b_{\gamma_0})f = \rho_{\gamma_0}(a_{\gamma_0})\rho_{\gamma_0}(b_{\gamma_0})f$.
4. $\forall a_{\gamma_0} : \rho_{\gamma_0}(a^*_{\gamma_0})f = \rho_{\gamma_0}(a_{\gamma_0})^*f$.

Before showing those please recall first, how the inductive limit of operators $\lim_\to[\pi_\gamma(\phi_{\gamma\gamma_0}a_{\gamma_0})]$ is given on $\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$:

$$\forall a_{\gamma_0} \in \mathfrak{U}_{\gamma_0} : \rho_{\gamma_0}(a_{\gamma_0}) : \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma) \to \lim_\to \mathcal{H}_\gamma, f \mapsto \begin{cases} \psi_\gamma\pi_\gamma(\phi_{\gamma\gamma_0}a_{\gamma_0})\psi_\gamma^{-1}f & \text{if } f \in \mathrm{im}(\phi_\gamma) \wedge \gamma \geq \gamma_0 \\ 0 & \text{else} \end{cases}$$

If $f \in \mathrm{im}(\psi_\gamma)$ but $\gamma \geq \gamma_0$ does not hold, then 2.) - 4.) are satisfied trivially. Now let $f \in \mathrm{im}(\psi_\gamma)$ with $\gamma \geq \gamma_0$. We then have that 1.) follows directly since ϕ and π are $*$-morphisms and ψ is linear. 2.) follows by inserting $id. = \psi_\gamma^{-1}\psi_\gamma$ and 3.) follows, since

$\psi_\gamma^* = \psi_\gamma^{-1}$ holds (since ψ_γ is an isometry). Explicitly:

$$\begin{aligned}\rho_{\gamma_0}(\alpha a_{\gamma_0} + \beta b_{\gamma_0})f &= \psi_\gamma \pi_\gamma(\phi_{\gamma\gamma_0}(\alpha a_{\gamma_0} + \beta b_{\gamma_0}))\psi_\gamma^{-1} f \\ &= \alpha \psi_\gamma \pi_\gamma(\phi_{\gamma\gamma_0}(a_{\gamma_0}))\psi_\gamma^{-1} f + \beta \psi_\gamma \pi_\gamma(\phi_{\gamma\gamma_0} b_{\gamma_0})\psi_\gamma^{-1} f \\ &= \alpha \rho_{\gamma_0}(a_{\gamma_0}) f + \beta \rho_{\gamma_0}(b_{\gamma_0}) f\end{aligned}$$

and

$$\begin{aligned}\rho_{\gamma_0}(a_{\gamma_0} b_{\gamma_0})f &= \psi_\gamma \pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0} b_{\gamma_0})\psi_\gamma^{-1} f \\ &= \psi_\gamma \pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0})\psi_\gamma^{-1} \psi_\gamma \pi_\gamma(\phi_{\gamma\gamma_0} b_{\gamma_0})\psi_\gamma f \\ &= \rho_{\gamma_0}(a_{\gamma_0})\rho_{\gamma_0}(b_{\gamma_0}) f\end{aligned}$$

and finally:

$$\begin{aligned}\rho_{\gamma_0}(a_{\gamma_0}^*)f &= \psi_\gamma \pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0}^*)\psi_\gamma^{-1} f \\ &= \psi_\gamma \pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0})^* \psi_\gamma^{-1} f \\ &= \left(\psi_\gamma \pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0})\psi_\gamma^{-1}\right)^* f \\ &= \rho_{\gamma_0}(a_{\gamma_0})^* f\end{aligned}$$

By the last assertion this gives a representation on all $\lim_\rightarrow \mathcal{H}_\gamma$.
Hence we have a family of representations on $\lim_\rightarrow \mathcal{H}_\gamma$. Now we want to use this to obtain a representation of $\lim_\rightarrow \mathfrak{U}_\gamma$ on $\lim_\rightarrow \mathcal{H}_\gamma$.

Assertion 4: Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{U}$ be a C^*-algebra with dense $*$-subalgebra $V \subset \mathfrak{U}$. Let π be a representation of V on $\mathcal{H}$. Then this extends to a unique representation of $\mathfrak{U}$ on $\mathcal{H}$ by defining for $(a_n)_{n\in\mathbb{N}} \subset V$ with $a_n \to a$:

$$\pi(a) := \lim_{n\to\infty} \pi(a_n)$$

Where the limit is with respect the usual operator norm.

Proof of Assertion 4: Recall, that the algebra of bounded operators on a Hilbert space $\mathcal{B}(\mathcal{H})$ together with the operator norm forms a C^*-algebra (recall example 1). Now a representation is just a $*$-morphism from a $*$-algebra to the C^*-algebra of bounded operators on a Hilbert space. Hence the claim follows with lemma 9 since representations are bounded by corollar 1.

Assertion 5: The representation obtained by extending

$$\lim_{\rightarrow} \pi_\gamma : \bigcup_{\gamma\in L} \operatorname{im}(\psi_\gamma) \to \mathcal{B}(\lim_{\rightarrow} \mathcal{H}_\gamma), a \in \operatorname{im}(\psi_{\gamma_0}) \mapsto \rho_{\gamma_0}(\phi_{\gamma_0}^{-1} a)$$

using the result of Assertion 4 is an inductive limit of the inductive family of representations $(\pi_\gamma)_{\gamma\in L}$.

Proof of Assertion 5: We show that for all $a_{\gamma_0} \in \mathfrak{U}_{\gamma_0}$ the following holds:

$$(\lim_{\to} \pi_\gamma)(\phi_{\gamma_0} a_{\gamma_0})\psi_{\gamma_0} = \psi_{\gamma_0}\pi_{\gamma_0}(a_{\gamma_0})$$

Therefore observe:

$$\begin{aligned}(\lim_{\to} \pi_\gamma)(\phi_{\gamma_0} a_{\gamma_0})\psi_{\gamma_0} &= \rho_{\gamma_0}(\phi_{\gamma_0}^{-1}\phi_{\gamma_0} a_{\gamma_0})\psi_{\gamma_0} \\ &= \rho_{\gamma_0}(a_{\gamma_0})\psi_{\gamma_0} \\ &= \lim_{\to}[\pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0})]\psi_{\gamma_0} \\ &= \psi_{\gamma_0}\pi_{\gamma_0}(a_{\gamma_0})\end{aligned}$$

Here we have used, that $\lim_{\to}[\pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0})]$ is the inductive limit of the inductive family of bounded operators $(\pi_\gamma(\phi_{\gamma\gamma_0} a_{\gamma_0}))_{\gamma\in L}$. Hence $(\lim_{\to} \pi_\gamma)$ is an inductive limit of the inductive family of representations $(\pi_\gamma)_{\gamma\in L}$.

Assertion 6: The inductive limit representation is unique.

Proof of Assertion 6: Assume, that there is a representation

$$\sigma : \lim_{\to} \mathfrak{U}_\gamma \to \mathcal{B}(\lim_{\to} \mathcal{H}_\gamma)$$

such that for all $a_\gamma \in \mathfrak{U}_\gamma$ we have:

$$\sigma(\phi_\gamma a_\gamma)\psi_\gamma = \psi_\gamma \pi_\gamma(a_\gamma) \tag{3.10}$$

Assume further $\sigma \neq \lim_{\to} \pi_\gamma$, i.e. there exists a $f \in \lim_{\to} \mathcal{H}_\gamma$ and a $a \in \lim_{\to} \mathfrak{U}_\gamma$ with $\sigma(a)f \neq (\lim_{\to} \pi_\gamma)(a)f$. But by eq. (3.10) we have, that σ and $\lim_{\to} \pi_\gamma$ must be the same on $\bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma) \subset \lim_{\to} \mathfrak{U}_\gamma$ and $\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma) \subset \lim_{\to} \mathcal{H}_\gamma$. Hence by Assertion 4 and Assertion 2, they must be same on all $\lim_{\to} \mathcal{H}_\gamma$ and $\lim_{\to} \mathfrak{U}_\gamma$. □

We now want to understand how the property that an element of a Hilbert space is cyclic behaves under inductive limits:

Lemma 16 (Cyclic vectors and inductive limits)
Let $(\mathcal{H}_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive family of Hilbert spaces with inductive limit $(\lim_{\to} \mathcal{H}_\gamma, (\psi_\gamma)_{\gamma\in L})$. Let $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive family of C^*-algebras with inductive limit $(\lim_{\to} \mathfrak{U}_\gamma, (\phi_\gamma)_{\gamma\in L})$. Let $(\pi_\gamma)_{\gamma\in L}$ be a corresponding inductive family of representations with inductive limit $\lim_{\to} \pi_\gamma$. Let further $(\Omega_\gamma)_{\gamma\in L}$ be a family of vectors $\Omega_\gamma \in \mathcal{H}_\gamma$ such that for each $\gamma \in L$ it holds, that Ω_γ is a cyclic vector for π_γ, i.e. $\pi_\gamma(\mathfrak{U}_\gamma)\Omega_\gamma$ is dense in $\mathcal{H}_\gamma$ and let further

$$\psi_{\gamma'\gamma}\Omega_\gamma = \Omega_{\gamma'}$$

hold for all $\gamma' \geq \gamma \in L$. Then: $\lim_{\to} \Omega_\gamma := \psi_\gamma\Omega_\gamma \in \lim_{\to} \mathcal{H}_\gamma$ is well defined and cyclic for $\lim_{\to} \pi_\gamma$.

Proof. We first show, that $\lim_{\to} \Omega_\gamma$ is well defined. Therefore observe:

$$\begin{aligned}\lim_{\to} \Omega_\gamma &= \psi_\gamma\Omega_\gamma \\ &= \psi_{\gamma'}\psi_{\gamma'\gamma}\Omega_\gamma \\ &= \psi_{\gamma'}\Omega_{\gamma'}\end{aligned}$$

We now show, that $[\lim_\rightarrow \pi_\gamma] (\lim_\rightarrow \mathfrak{U}_\gamma)\Omega \subset \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$ is dense in $\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$. Then the claim follows, since denseness is transitive.

Therefore let $f \in \bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$. I.e., there is a $\gamma \in L$ and there is a $f_\gamma \in \mathcal{H}_\gamma$ with $\psi_\gamma(f_\gamma) = f$. Now we have, that Ω_γ is cyclic vector of π_γ in $\mathcal{H}_\gamma$, i.e. there is a sequence $(a_n^\gamma)_{n\in\mathbb{N}} \subset \mathfrak{U}_\gamma$ s.th.:

$$\lim_{n\to\infty} \pi_\gamma \left(a_n^{(\gamma)}\right) \Omega_\gamma = f_\gamma$$

Hence we have:

$$\psi_\gamma \left(\lim_{n\to\infty} \pi_\gamma \left(a_n^{(\gamma)}\right) \Omega_\gamma\right) = f$$

Now ψ_γ is bounded and continuous as an isometry. Hence we have:

$$\lim_{n\to\infty} \psi_\gamma \left(\pi_\gamma \left(a_n^{(\gamma)}\right) \Omega_\gamma\right) = f$$

Now we have:

$$\begin{aligned}\psi_\gamma \left(\pi_\gamma \left(a_n^{(\gamma)}\right) \Omega_\gamma\right) &= \psi_\gamma \circ \pi_\gamma \left(a_n^{(\gamma)}\right) \circ \psi_\gamma^{-1}\psi_\gamma(\Omega_\gamma) \\ &= \rho_\gamma \left(a_n^{(\gamma)}\right) \Omega\end{aligned}$$

Here we used, that $\phi_\gamma\Omega_\gamma = \Omega$ and further used the definition of ρ_γ from the proof of theorem 24.

Now observe, that we can define $a_n := \phi_\gamma\left(a_n^\gamma\right) \in \mathrm{im}(\phi_\gamma) \subset \lim_\rightarrow \mathfrak{U}_\gamma$ and hence we can write:

$$\begin{aligned}\rho_\gamma \left(a_n^{(\gamma)}\right) \Omega &= \rho_\gamma \left(\phi_\gamma^{-1} a_n\right) \Omega \\ &= \left[\lim_\rightarrow \pi_\gamma\right] (a_n)\Omega\end{aligned}$$

and hence we have constructed for each $f \in \bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma)$ a sequence $(a_n)_{n\in\mathbb{N}} \subset \lim_\rightarrow \mathfrak{U}_\gamma$ with :

$$\lim_{n\to\infty} \left[\lim_\rightarrow \pi_\gamma\right] (a_n)\Omega = f$$

And hence the claim follows, since denseness is transitive and $\bigcup_{\gamma\in L} \mathrm{im}(\psi_\gamma)$ is dense in $\lim_\rightarrow \mathcal{H}_\gamma$. □

With this we can define the inductive limit of cyclic representations:

Definition 21
Let $(\mathcal{H}_\gamma, \psi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive family of Hilbert spaces with inductive limit $(\lim_\rightarrow \mathcal{H}_\gamma, (\psi_\gamma)_{\gamma\in L})$. Let $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive family of C^*-algebras with inductive limit $(\lim_\rightarrow \mathfrak{U}_\gamma, (\phi_\gamma)_{\gamma\in L})$. Then:

1. *An inductive family of cyclic representations is a family $(\pi_\gamma, \Omega_\gamma)_{\gamma\in L}$ such that $(\pi_\gamma)_{\gamma\in L}$ is an inductive family of representations and Ω_γ is a corresponding inductive family of cyclic vectors.*

2. *Its inductive limit is given by the tuple $(\lim_\rightarrow \pi_\gamma, \lim_\rightarrow \Omega_\gamma)$, which is again a cyclic representation by the last theorem.*

We now want to understand, how the Gel'fand transform behaves under inductive limits:

Lemma 17
Let $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ be an inductive family of abelian, unital C^*-algebras and let $\left(\Delta(\mathfrak{U}_\gamma), p_{\gamma\gamma'} = \phi^*_{\gamma'\gamma}\right)_{\gamma',\gamma\in L}$ be the corresponding projective family of compact Hausdorff spaces given by theorem 18. Let $\left(C(\Delta(\mathfrak{U}_\gamma), \psi_{\gamma'\gamma} = p^*_{\gamma\gamma'}\right)_{\gamma,\gamma'\in L}$ be the inductive family of the algebras of continuous functions on the spectra given by theorem 20. Let $\mathcal{G}_\gamma : \mathfrak{U}_\gamma \to C(\Delta(\mathfrak{U}_\gamma))$ and $\mathcal{G} : \lim_\to \mathfrak{U}_\gamma \to C(\lim_\leftarrow \Delta(\mathfrak{U}_\gamma))$ be the corresponding Gel'fand transforms, which we denote in both cases also by $\check{\cdot}$. Then:

1. $\psi_{\gamma'\gamma} \circ \mathcal{G}_\gamma = \mathcal{G}_{\gamma'} \circ \phi_{\gamma'\gamma}$
2. $\psi_\gamma \circ \mathcal{G}_\gamma = \mathcal{G} \circ \phi_\gamma$.

Proof. 1.) Let $a_\gamma \in \mathfrak{U}_\gamma$ and $\chi_{\gamma'} \in \Delta(\mathfrak{U}_{\gamma'})$. Then:

$$\begin{aligned}\left(\psi_{\gamma'\gamma} \circ \mathcal{G}_\gamma(a_\gamma)\right)(\chi_{\gamma'}) &= \check{a}_\gamma \circ p_{\gamma\gamma'}(\chi_{\gamma'})\\ &= \check{a}_\gamma\left(p_{\gamma\gamma'}\chi_{\gamma'}\right)\\ &= \left(p_{\gamma\gamma'}\chi_{\gamma'}\right)(a_\gamma)\\ &= \chi_{\gamma'} \circ \phi_{\gamma'\gamma}(a_\gamma)\\ &= \left(\mathcal{G}_{\gamma'} \circ \phi_{\gamma'\gamma}(a_\gamma)\right)(\chi_{\gamma'})\end{aligned}$$

and hence $\psi_{\gamma'\gamma} \circ \mathcal{G}_\gamma = \mathcal{G}_{\gamma'} \circ \phi_{\gamma'\gamma}$.
2.) Let $a_\gamma \in \mathfrak{U}_\gamma$ and $\chi \in \Delta(\lim_\to \mathfrak{U}_\gamma)$. Then:

$$\begin{aligned}\left(\psi_\gamma \circ \mathcal{G}_\gamma(a_\gamma)\right)(\chi) &= (\check{a}_\gamma \circ p_\gamma)(\chi)\\ &= \check{a}_\gamma(p_\gamma\chi)\\ &= p_\gamma\chi(a_\gamma)\\ &= \chi \circ \phi_\gamma(a_\gamma)\\ &= \chi(\phi_\gamma a_\gamma)\\ &= \left(\mathcal{G} \circ \phi_\gamma(a_\gamma)\right)(\chi)\end{aligned}$$

and hence $\psi_\gamma \circ \mathcal{G}_\gamma = \mathcal{G} \circ \phi_\gamma$. □

With this Lemma we now investigate the case of Gel'fand representations:

Theorem 25
Let $(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ be an inductive family of abelian, unital C^*-algebras and let $(\Delta(\mathfrak{U}_\gamma), p_{\gamma\gamma'} = \phi^*_{\gamma'\gamma})_{\gamma',\gamma\in L}$ be the corresponding projective family of compact Hausdorff spaces given by theorem 18. Let further $(d\mu_\gamma)_{\gamma\in L}$ be a family of regular, Borel probability measures such that

$$(\Delta(\mathfrak{U}_\gamma), d\mu_\gamma, p_{\gamma',\gamma})_{\gamma,\gamma'\in L}$$

becomes a projective family of measure spaces. Let

$$(\mathcal{H}_\gamma = L^2(\Delta(\mathfrak{U}_\gamma), d\mu_\gamma), \psi_{\gamma'\gamma} = p^*_{\gamma\gamma'})_{\gamma,\gamma'\in L}$$

be the corresponding inductive family of Hilbert spaces obtained by theorem 22. Then:

1. *The family $(\pi_\gamma)_{\gamma\in L}$ with*

$$\pi_\gamma : \mathfrak{U}_\gamma \to \mathcal{B}(\mathcal{H}_\gamma), a_\gamma \mapsto (\pi(a_\gamma) : \mathcal{H}_\gamma \to \mathcal{H}_\gamma, f_\gamma \mapsto \check{a}_\gamma f)$$

   where $\check{\cdot}$ denotes the Gel'fand transformation on $\mathfrak{U}_\gamma$ is an inductive family of representations.

2. *Its inductive limit equals the representation*

$$\pi : \lim_{\rightarrow} \mathfrak{U}_\gamma \to \mathcal{B}(\lim_{\rightarrow} \mathcal{H}_\gamma), a \mapsto \left(\pi(a) : \lim_{\rightarrow} \mathcal{H}_\gamma \to \lim_{\rightarrow} \mathcal{H}_\gamma, f \mapsto \check{a} f\right)$$

where $\check{}$ *denotes the Gel'fand transformation on* $\lim_\rightarrow \mathfrak{U}_\gamma$.

Proof. 1.) Therefore we have to show:

1. $\forall \gamma \in L : \pi_\gamma$ is representation on $\mathcal{H}_\gamma$.
2. $\forall \gamma' \geq \gamma \in L : \pi_{\gamma'}(\phi_{\gamma'\gamma} a_\gamma) \circ \psi_{\gamma'\gamma} = \psi_{\gamma'\gamma} \circ \pi_\gamma(a_\gamma)$.

The first was already shown lemma 4. We now show the second. Therefore we first show, that this relation holds on $C^\infty(\Delta(\mathfrak{U}_\gamma))$. Then by denseness arguments the relation extends to all of $L^2(\Delta(\mathfrak{U}_\gamma), d\mu_\gamma)$.

Let $f \in C^\infty(\Delta(\mathfrak{U}_\gamma))$. We first recall some definitions. First recall, how the projections on the projective family of measure spaces $(\Delta(\mathfrak{U}_\gamma), p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ are defined:

$$\forall \gamma' \geq \gamma : p_{\gamma\gamma'} : \Delta(\mathfrak{U}_{\gamma'}) \to \Delta(\mathfrak{U}_\gamma), \chi_{\gamma'} \mapsto \chi_{\gamma'} \circ \phi_{\gamma'\gamma} = \phi^*_{\gamma'\gamma}\chi_{\gamma'}$$

Now recall, how the isometries $\psi_{\gamma'\gamma}$ of the inductive family $(L^2(\Delta(\mathfrak{U}_\gamma), d\mu_\gamma), \psi_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ are defined:

$$\forall \gamma' \geq \gamma : \psi_{\gamma'\gamma} : L^2(\Delta(\mathfrak{U}_\gamma)) \to L^2(\Delta(\mathfrak{U}_{\gamma'})), f \mapsto p^*_{\gamma\gamma'} f = f \circ p_{\gamma\gamma'} = f \circ \phi^*_{\gamma'\gamma}$$

I.e. for a character $\chi_{\gamma'}$ and $f \in C^\infty(\Delta(\mathfrak{U}_\gamma)) \subset L^2(\Delta(\mathfrak{U}_\gamma), d\mu_\gamma)$ we have

$$\begin{aligned}\left(\psi_{\gamma'\gamma} f\right)(\chi_{\gamma'}) &= \left(f \circ \phi^*_{\gamma'\gamma}\right)(\chi_{\gamma'}) \\ &= f\left(\chi_{\gamma'} \circ \phi_{\gamma'\gamma}\right)\end{aligned}$$

With this and theorem 25 we have now:

$$\begin{aligned}\left(\pi_{\gamma'}(\phi_{\gamma'\gamma} a_\gamma)\psi_{\gamma'\gamma} f\right)(\chi_{\gamma'}) &= \left(\mathcal{G}_{\gamma'} \circ \phi_{\gamma'\gamma}(a_\gamma)\right) \cdot \psi_{\gamma'\gamma} f \\ &= \left(\psi_{\gamma'\gamma} \circ \mathcal{G}_\gamma(a_\gamma)\right) \cdot \psi_{\gamma'\gamma} f \\ &= \psi_{\gamma'\gamma}\left(\mathcal{G}_\gamma(a_\gamma) \cdot f\right) \\ &= \psi_{\gamma'\gamma}\left(\pi_\gamma(a_\gamma) f\right)\end{aligned}$$

and hence the identity holds pointwise on $C^\infty(\Delta(\mathfrak{U}_\gamma))$ and hence by denseness arguments on $\mathcal{H}_\gamma$.
2.) We want to show, that the inductive limit representation equals the Gel'fand representation π on the inductive limit. Therefore we first show, that the inductive limit representation restricted to $\bigcup_{\gamma \in L} \operatorname{im}(\phi_\gamma) \subset \lim_\rightarrow \mathfrak{U}_\gamma$ on $\bigcup_{\gamma \in L} \operatorname{im}(\psi_\gamma) \subset \lim_\rightarrow \mathcal{H}_\gamma$ equals the Gel'fand representation π thereon. Then we use the Assertions 2 and 4 proven in theorem 24 to conclude, that it equals the Gel'fand representation π on all of $\lim_\rightarrow \mathcal{H}_\gamma$.

Let $a \in \bigcup_{\gamma \in L} \operatorname{im}(\phi_\gamma)$ with $a = \phi_{\gamma_0}(a_{\gamma_0})$ and $f \in \bigcup_{\gamma \in L} \operatorname{im}(\psi_\gamma)$ with $f = \psi_\gamma(f_\gamma)$. We then have further, that there is a $\hat{\gamma} \geq \gamma, \gamma_0$ and we have further $\phi_{\gamma_0} = \phi_{\hat{\gamma}} \circ \phi_{\hat{\gamma}\gamma_0}$ and $\psi_\gamma = \psi_{\hat{\gamma}} \circ \psi_{\hat{\gamma}\gamma}$. We then have:

$$\begin{aligned}\left[\lim_{\rightarrow} \pi_\gamma\right](a) f &= \left[\lim_{\rightarrow} \pi_\gamma\right](\phi_{\gamma_0} a_{\gamma_0})\psi_\gamma f_\gamma \\ &= \left[\lim_{\rightarrow} \pi_\gamma\right](\phi_{\hat{\gamma}} \circ \phi_{\hat{\gamma}\gamma_0} a_{\gamma_0})(\psi_{\hat{\gamma}} \circ \psi_{\hat{\gamma}\gamma} f_\gamma) \\ &= \left[\lim_{\rightarrow} \pi_\gamma\right](\phi_{\hat{\gamma}} a_{\hat{\gamma}})(\psi_{\hat{\gamma}} f_{\hat{\gamma}}) \\ &= \psi_{\hat{\gamma}} \pi_{\hat{\gamma}}(a_{\hat{\gamma}}) f_{\hat{\gamma}}\end{aligned}$$

Where we have set $a_{\hat{\gamma}} = \phi_{\hat{\gamma}\gamma_0} a_{\gamma_0}$, $f_{\hat{\gamma}} = \psi_{\hat{\gamma}\gamma} f_\gamma$. Further in the last line we have used the defining property of the inductive limit representation. Now we have by definition of $\pi_{\hat{\gamma}}$:

$$\psi_{\hat{\gamma}} \pi_{\hat{\gamma}}(a_{\hat{\gamma}}) f_{\hat{\gamma}} = \psi_{\hat{\gamma}} \left(\check{a}_{\hat{\gamma}} \cdot f_{\hat{\gamma}} \right)$$

Now we have to show two things:

1. For $f_{\hat{\gamma}} \in L^2(\Delta(\mathfrak{U}_\gamma), d\mu_\gamma)$ and $g_{\hat{\gamma}} \in C^\infty(\Delta(\mathfrak{U}_\gamma))$ it holds, that: $\psi_{\hat{\gamma}}(g_{\hat{\gamma}} \cdot f_{\hat{\gamma}}) = \psi_{\hat{\gamma}}(g_{\hat{\gamma}}) \cdot \psi_{\hat{\gamma}}(f_{\hat{\gamma}})$.
2. For $a_{\hat{\gamma}} \in \mathfrak{U}_{\hat{\gamma}}$ it holds, that $\psi_{\hat{\gamma}} \circ \mathcal{G}(a_{\hat{\gamma}}) = \mathcal{G} \circ \phi_{\hat{\gamma}}(a_{\hat{\gamma}})$.

Therefore recall first, that by theorem 22 we have $\lim_\rightarrow \mathcal{H}_\gamma \cong L^2(\Delta(\lim_\rightarrow \mathfrak{U}_\gamma), d\mu)$ and further we have:

$$\psi_{\hat{\gamma}} : \mathcal{H}_{\hat{\gamma}} \rightarrow L^2(\Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma), d\mu), f_{\hat{\gamma}} \mapsto \psi_{\hat{\gamma}} f_{\hat{\gamma}} = p^*_{\hat{\gamma}} f_\gamma$$

Further we have by theorem 18 that $\lim_\leftarrow \Delta(\mathfrak{U}_\gamma) \cong \Delta(\lim_\rightarrow \mathfrak{U}_\gamma)$ and further:

$$p_{\hat{\gamma}} : \Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma) \rightarrow \Delta(\mathfrak{U}_\gamma), \chi \mapsto \psi^*_{\hat{\gamma}} \chi$$

Hence we have first:

$$\begin{aligned} \psi_{\hat{\gamma}}(g_{\hat{\gamma}} \cdot f_{\hat{\gamma}}) &= (g_{\hat{\gamma}} \cdot f_{\hat{\gamma}}) \circ p_{\hat{\gamma}} \\ &= (g_{\hat{\gamma}} \circ p_{\hat{\gamma}}) \cdot (f_{\hat{\gamma}} \circ p_{\hat{\gamma}}) \\ &= \psi_{\hat{\gamma}}(g_{\hat{\gamma}}) \cdot \psi_{\hat{\gamma}}(f_{\hat{\gamma}}) \end{aligned}$$

And hence the first equation holds. Further the second equation was already proven in theorem 25. We now have shown, that the inductive limit representation equals the representation π on a dense subset of $\lim_\rightarrow \mathfrak{U}_\gamma$ and on a dense subset of $\lim_\rightarrow \mathcal{H}_\gamma$. Now by Assertion 2 and Assertion 4 proven in the proof of definition 20 this extends uniquely to a representation on all $\lim_\rightarrow \mathcal{H}_\gamma$. Hence the claim is shown. □

3.7. Spectral Theorem for Inductive Limit C^*-Algebras

We first want to prove the following structure theorem regarding cyclic representations of inductive limit C^*-algebras. Recalling the statement of lemma 4 the basic statement of the following is, that any cyclic representation of an inductive limit C^*-algeba arises as an inductive limit of cyclic representations.

Theorem 26
Let $(\mathfrak{U}_\gamma, \phi_{\gamma',\gamma})$ be an inductive family of abelian, unital C^*-algebras with inductive limit $(\lim_\rightarrow \mathfrak{U}_\gamma, (\phi_\gamma)_{\gamma \in L})$. Let ω be a state of $\mathfrak{U}$. Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the cyclic representation of $\lim_\rightarrow \mathfrak{U}_\gamma$ corresponding to the state ω given by lemma 4 and let further $(\mathcal{H}_{\omega_\gamma}, \pi_{\omega_\gamma}, \Omega_{\omega_\gamma})$ be the cyclic representation of $\mathfrak{U}_\gamma$ corresponding to the state $\omega_\gamma := \omega \circ \phi_\gamma$ given by lemma 4. Then: The representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is the inductive limit of the inductive family of cyclic representations $(\mathcal{H}_{\omega_\gamma}, \pi_{\omega_\gamma}, \Omega_{\omega_\gamma})_{\gamma,\gamma' \in L}$.

Proof. Please recall first from lemma 4:

$$\begin{aligned} \mathcal{H}_\omega &= L^2(\Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma), d\mu) \\ \mathcal{H}_{\omega_\gamma} &= L^2(\Delta(\mathfrak{U}_\gamma), d\mu_\gamma) \end{aligned}$$

Here the measures are via theorem 7 uniquely characterized by:

$$\forall b \in \lim_{\rightarrow} \mathfrak{U}_\gamma : \int_{\Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma)} \check{b} d\mu = \omega(b^* b)$$
$$\forall b_\gamma \in \mathfrak{U}_\gamma : \int_{\Delta(\mathfrak{U}_\gamma)} \check{b_\gamma} d\mu_\gamma = \omega_\gamma(b_\gamma^* b_\gamma)$$

and further it holds, that

$$\Omega_\omega = \mathbb{1} \in L^2(\Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma), d\mu)$$
$$\Omega_{\omega_\gamma} = \mathbb{1} \in L^2(\Delta(\mathfrak{U}_\gamma), d\mu_\gamma)$$

and finally

$$\forall b \in \lim_{\rightarrow} \mathfrak{U}_\gamma : \pi_\omega(b) : f \in \mathcal{H}_\omega \mapsto \check{b} \cdot f \in \mathcal{H}_\omega$$
$$\forall b_\gamma \in \mathfrak{U}_\gamma : \pi_{\omega_\gamma}(b_\gamma) : f_\gamma \in \mathcal{H}_{\omega_\gamma} \mapsto \check{b}_\gamma \in \mathcal{H}_{\omega_\gamma}$$

Please recall further, that it was already shown in theorem 18, that $(\Delta(\mathfrak{U}_\gamma), p_{\gamma\gamma'} = \phi^*_{\gamma'\gamma})_{\gamma,\gamma' \in L}$ is a projective family of compact Hausdorff spaces. Further recall from the same theorem, that $\lim_{\leftarrow} \Delta(\mathfrak{U}_\gamma) = \Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma)$ and $p_\gamma = \phi^*_\gamma$ under this identification. We now have to show that $d\mu_\gamma$ really defines a projective family of measures with projective limit measure $d\mu$, and that Ω_{ω_γ} is an inductive family of cyclic vectors with inductive limit Ω_ω.

Assertion 1: $\left(\Delta(\mathfrak{U}_\gamma), \mu_\gamma), p_{\gamma'\gamma}\right)_{\gamma,\gamma' \in L}$ is a projective family of measure spaces with projective limit $((\Delta(\lim_{\rightarrow} \mathfrak{U}_\gamma), d\mu), (p_\gamma)_{\gamma \in L})$.

Proof of Assertion 1: For this we have to show, that $\forall \gamma' \geq \gamma : p_{\gamma\gamma' *}\mu_{\gamma'} = \mu_\gamma$. Therefore observe first, that for $\gamma' \in L$ the measure $\mu_{\gamma'}$ is defined by:

$$\forall b_{\gamma'} \in \mathfrak{U}_{\gamma'} : \int_{\Delta(\mathfrak{U}_{\gamma'})} \check{b}_{\gamma'} d\mu_{\gamma'} = \omega_{\gamma'}(b^*_{\gamma'} b_{\gamma'})$$

Now let $b_{\gamma'} \in \operatorname{im}(\phi_{\gamma'\gamma})$ with $b_{\gamma'} = \phi_{\gamma'\gamma} b_\gamma$. Then we have:

$$\omega_{\gamma'}(b^*_{\gamma'} b_{\gamma'}) = \omega_{\gamma'}(\phi_{\gamma'\gamma}(b^*_\gamma b_\gamma)) = \omega_{\gamma'} \circ \phi_{\gamma'\gamma}(b^*_\gamma b_\gamma) = \omega_\gamma(b^*_\gamma b_\gamma)$$

Now consider the left hand side. Therefore recall first, that we have shown in the proof of lemma 17, that

$$\mathcal{G}_{\gamma'} \circ \phi_{\gamma'\gamma} b_\gamma = \check{b}_\gamma \circ p_{\gamma\gamma'}$$

holds. With this we have for the left hand side:

$$\int_{\Delta(\mathfrak{U}_{\gamma'})} \check{b}_{\gamma'} d\mu_{\gamma'} = \int_{\Delta(\mathfrak{U}_{\gamma'})} \check{b}_\gamma \circ p_{\gamma\gamma'} d\mu_{\gamma'}$$
$$= \int_{\Delta(\mathfrak{U}_\gamma)} \check{b}_\gamma d(p_{\gamma\gamma' *}\mu)$$

where we have used the change of variables formula for pushforward measures. Hence we have:

$$\forall b_\gamma \in \mathfrak{U}_\gamma : \int_{\Delta(\mathfrak{U}_\gamma)} \check{b}_\gamma d(p_{\gamma\gamma' *}\mu) = \omega_\gamma(b^*_\gamma b_\gamma)$$

and since this characterizes the measure uniquely by theorem 7, $\mu_\gamma = p_{\gamma\gamma'*}\mu_{\gamma'}$ follows.

Now we have to show $p_{\gamma*}\mu = \mu_\gamma$. By uniqueness of the projective limit measure, then the assertion follows. Therefore recall first, that we have shown in lemma 17, that:

$$\mathcal{G} \circ \phi_\gamma b_\gamma = \check{b}_\gamma \circ p_\gamma$$

Now let $b \in \mathrm{im}(\phi_\gamma)$ with $b = \phi_\gamma(b_\gamma)$. We then have:

$$\int_{\Delta(\lim_\rightarrow \mathfrak{U}_\gamma)} \check{b} d\mu = \omega(b^* b)$$

We now have for the right hand side:

$$\omega(b^*b) = \omega(\psi_\gamma(b_\gamma^* b_\gamma)) = \omega \circ \psi_\gamma(b_\gamma^* b_\gamma) = \omega_\gamma(b_\gamma^* b_\gamma)$$

For the left hand side we have:

$$\begin{aligned}\int_{\Delta(\lim_\rightarrow \mathfrak{U}_\gamma)} \check{b} d\mu &= \int_{\Delta(\lim_\rightarrow \mathfrak{U}_\gamma)} \mathcal{G}(\psi_\gamma b_\gamma) d\mu \\ &= \int_{\Delta(\lim_\rightarrow \mathfrak{U}_\gamma)} \check{b}_\gamma \circ p_\gamma d\mu \\ &= \int_{\Delta(\mathfrak{U}_\gamma)} b_\gamma d(p_{\gamma*}\mu)\end{aligned}$$

And hence it follows, that:

$$\int_{\Delta(\mathfrak{U}_\gamma)} b_\gamma d(p_{\gamma*}\mu) = \omega_\gamma(b_\gamma^* b_\gamma)$$

and hence by theorem 7 $p_{\gamma*}\mu = \mu_\gamma$ follows. Since the projectvie limit of measure spaces is unique, the assertion follows.

Please observe, that with this and with theorem 22 it also follows, that $\mathcal{H}_{\omega_\gamma}$ is an inductive family of Hilbert spaces with inductive limit $\mathcal{H}_\omega$.

Assertion 2: $(\Omega_{\omega_\gamma})_{\gamma \in L}$ with $\Omega_{\omega_\gamma} = \mathbb{1} \in L^2(\Delta(\mathfrak{U}_\gamma), d\mu_\gamma)$ is an inductive family of vectors corresponding to the inductive family of Hilbert spaces $(\mathcal{H}_{\omega_\gamma}, \phi_{\gamma'\gamma} = p^*_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ with inductive limit $\lim_\rightarrow \Omega_{\omega_\gamma} = \Omega_\omega$.

Proof of Assertion 2: Please observe, that for each $\gamma \in L$ the function $\mathbb{1} : \Delta(\mathfrak{U}_\gamma) \to \mathbb{R}, \chi_\gamma \mapsto 1$ is well defined and lies in $C^\infty(\mathfrak{U}_\gamma)$. Now observe further, that for all $\chi \in \Delta(\mathfrak{U}_{\gamma'})$:

$$\left(\psi_{\gamma'\gamma}\Omega_{\omega_\gamma}\right)(\chi_{\gamma'}) = \left(\mathbb{1} \circ p_{\gamma\gamma'}\right)\chi_{\gamma'} = 1$$

and hence $\psi_{\gamma'\gamma}\Omega_{\omega_\gamma} = \Omega_{\omega_{\gamma'}}$. Now let $\chi \in \Delta(\lim_\rightarrow \mathfrak{U}_\gamma) = \lim_\leftarrow \Delta(\mathfrak{U}_\gamma)$. Then:

$$\begin{aligned}\left(\lim_\rightarrow \Omega_{\omega_\gamma}\right)(\chi) &= \left(\psi_\gamma \Omega_{\omega_\gamma}\right)(\chi) \\ &= \mathbb{1} \circ p_\gamma(\chi) \\ &= \mathbb{1}\end{aligned}$$

Hence we have $\lim_\rightarrow \Omega_{\omega_\gamma} = \Omega_\omega$. That both are cyclic follows with lemma 4.

Proof of the theorem: With Assertion 1 we have, that $\mathcal{H}_{\omega_\gamma}$ is an inductive family of Hilbert spaces with inductive limit $\mathcal{H}_\omega$. With theorem 25, Assertion 2 and lemma 4 we have, that the cyclic representations $(\mathcal{H}_{\omega_\gamma}, \pi_{\omega_\gamma}, \Omega_{\omega_\gamma})_{\gamma \in L}$ are an inductive family of representations corresponding to the inductive family $\mathfrak{U}_\gamma$ of C^*-algebras. With this it follows further, that its inductive limit representation is given by the representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$. Hence the claim is shown. □

At this stage it is worth to notice, that this theorem does not answer the question, if a cyclic representation of a C^*-algebra can arise as an inductive limit of non-cyclic representations. That this situation is an interesting one, can be seen by considering a corresponding physical interpretation: The cyclic vector can be interpreted as the vacuum vector of a theory while the members of the inductive family could be interpreted as subsystems of the full physical system. Now the statement that a cyclic representation is an inductive limit of non-cyclic ones would mean in this case, that there exists a vacuum vector for the full system, which is no vacuum for any subsystem. Unfortunately the author of this thesis had not enough time to think on this situation, but he hopes to do so in future.

Using theorem 26 and theorem 4 we directly obtain the following:

Theorem 27
Let $(\mathfrak{U}_\gamma, \phi_{\gamma',\gamma})_{\gamma,\gamma' \in L}$ be an inductive family of abelian, unital C^*-algebras with inductive limit $(\mathfrak{U} = \lim_\rightarrow \mathfrak{U}_\gamma, (\phi_\gamma)_{\gamma \in L})$. Then each representation of $\mathfrak{U}$ can be written as a direct sum of the direct limits of the cyclic representations $(\mathcal{H}_{\omega_\gamma}, \pi_{\omega_\gamma}, \Omega_{\omega_\gamma})$ of $\mathfrak{U}_\gamma$ given by lemma 4 corresponding to the projective family of states $\omega_\gamma := \omega \circ \psi_\gamma$.

4. Spectral Theory of C^*-Dynamical Systems

In this section the spectral theory of C^*-dynamical systems is investigated. In the first section group actions on compact Hausdorff spaces and group actions on C^*-algebras (called C^*-dynamical systems) are introduced. Further the notions of fixed point C^*-subalgebras and quotient spaces of compact Hausdorff spaces under group actions are introduced. The next section investigates in which sense the spectrum of the fixed point C^*-subalgebra is related to the spectrum of the full algebra. The next section analyzes the compatibility of group actions on compact Hausdorff spaces with projective limits. The fourth section introduces inductive families of C^*-dynamical systems and further explores the case of the C^*-algebra of continuous functions in this context. In the last section the spectral theory of inductive limits of C^*-dynamical systems is investigated and it is shown that the duality between fixed point subalgebras and quotient spaces is compatible with inductive and projective limits.

The main references for the basic definitions of the first section are given by [10] and [37] for the C^*-side and [16] and [8] for the topological side. The theory of projective limits of group actions on compact Hausdorff spaces can be found in basic literature on loop quantum gravity, as [31]. The other results are mostly new. Especially the connection between fixed point C^*-subalgebras and quotient spaces and their relation to inductive and projective limits is firstly discussed in this thesis.

4.1. Group Actions and C^*-Dynamical Systems

We first define the notion of C^*-dynamical systems:

Definition 22 (C^*-dynamical system, cp. [10], [37])
A C^*-dynamical system is a tuple $(\mathfrak{U}, G, \Phi)$, such that $\mathfrak{U}$ is a C^*-algebra, G is a compact topological Hausdorff group G and Φ is a C^*-group action from the left, written as $\Phi : G \curvearrowright \mathfrak{U}$, which is a map

$$\Phi : G \times \mathfrak{U} \to \mathfrak{U}, (g, a) \mapsto \Phi_g(a)$$

such that for $\Phi : G \to \mathrm{Aut}(\mathfrak{U}), g \mapsto \Phi_g(\cdot)$ the following holds:

1. $\forall g, h \in G : \Phi_h \circ \Phi_g = \Phi_{hg}$.
2. $\forall g \in G : \Phi_{g^{-1}} = \Phi_g^{-1}$.
3. *Φ is continuous in pointwise norm topology on $\mathrm{Aut}(\mathfrak{U})$, i.e. $\forall a \in \mathfrak{U}$: $g \mapsto \|\Phi_g(a)\|$ is continuous.*
4. *$\forall g \in G : \Phi_g$ is an isometric $*$-morphism.*

In this case we write $\forall g \in G, \forall a \in \mathfrak{U} : \Phi(g, a) = \Phi_g(a) = ga$.

Please observe, that Φ_g is automatically an isometric $*$-automorphism since $\Phi_{g^{-1}}$ is an inverse for Φ_g. We now want to define the fixed point C^*-subalgebra of a C^*-dynamical system:

Definition 23 (cp. [2])
Let $(\mathfrak{U}, G, \Phi)$ be a C^*-dynamical system. Then define the fixed point C^*-subalgebra of $\mathfrak{U}$ as:

$$\mathrm{Fix}_G(\mathfrak{U}) := \{a \in \mathfrak{U} | \forall g \in G : ga = a\} \subset \mathfrak{U}$$

We now show, that the fixed point C^*-subalgebra is really a subalgebra:

Lemma 18
Let $(\mathfrak{U}, G, \Phi)$ be a C^*-dynamical system. Then $\mathrm{Fix}_G(\mathfrak{U}) \subset \mathfrak{U}$ is a C^*-subalgebra.

Proof. We first show, that it is closed as a subspace. Therefore let $(a_n)_{n\in\mathbb{N}} \subset \mathrm{Fix}_G(\mathfrak{U})$ be convergent in $\mathfrak{U}$. I.e. $\forall n \in \mathbb{N} : \forall g \in G : ga_n = a_n$ and $\lim_{n\to\infty} a_n \in \mathfrak{U}$. We have further, that $\Phi_g(\cdot)$ is an isometric $*$-morphism for all $g \in G$ and hence in particular continuous. Hence:

$$\Phi_g\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} \Phi_g(a_n) = \lim_{n\to\infty} a_n$$

And hence $\lim_{n\to\infty} a_n \in \mathrm{Fix}_G(\mathfrak{U})$. Now observe further, that $\mathrm{Fix}_G(\mathfrak{U})$ is closed under the $*$-algebra structure, since Φ_g is a $*$-morphism for each $g \in G$. □

We now want to investigate the case of abelian, unital C^*-algebras:

Lemma 19
Let $(\mathfrak{U}, G, \Phi)$ be a C^*-dynamical system with $\mathfrak{U}$ being abelian and unital.
Then: $\mathrm{Fix}_G(\mathfrak{U})$ is abelian and unital.

Proof. That $\mathrm{Fix}_G(\mathfrak{U})$ is abelian follows, since subalgebras of abelian C^*-algebras are abelian. It remains to show, that $\mathrm{Fix}_G(\mathfrak{U})$ is unital, i.e. that $\mathbb{1}_{\mathfrak{U}} \in \mathrm{Fix}_G(\mathfrak{U})$. Therefore recall first, that Φ_g is an $*$-automorphism for each $g \in G$ and hence unital by remark 2. Hence $\Phi(g, \mathbb{1}_{\mathfrak{U}}) = \mathbb{1}_{\mathfrak{U}}$ for all $g \in G$ and hence $\mathbb{1}_{\mathfrak{U}} \in \mathrm{Fix}_G(\mathfrak{U})$. □

We now define group actions on compact Hausdorff spaces:

Definition 24 (cp. [16])
Let X be a compact Hausdorff space and let G be a compact topological Hausdorff group. Then: A group action from the right on $\mathfrak{X}$, written as $\tilde{\Phi} : X \curvearrowleft G$, is a map

$$\tilde{\Phi} : X \times G \to X, (x, g) \mapsto \tilde{\Phi}_g(x)$$

which is continuous in product topology on $X \times G$ and further satisfies for $\tilde{\Phi}_g := \tilde{\Phi}_g(\cdot)$ the following:

1. $\forall g, h \in G : \tilde{\Phi}_h \circ \tilde{\Phi}_g = \tilde{\Phi}_{gh}$.
2. $\forall g \in G : \tilde{\Phi}_{g^{-1}} = \tilde{\Phi}_g^{-1}$

In this case we write $\forall g \in G, \forall x \in X : \tilde{\Phi}(x, g) = xg$.

In the following we will always assume, that G is a compact topological Hausdorff group. Please observe further, that $\tilde{\Phi}_g$ is automatically a homeomorphism, since $\tilde{\Phi}_{g^{-1}}$ is a continuous inverse fore $\tilde{\Phi}_g$. We now want to use this to define quotient spaces of compact Hausdorff spaces by group actions:

Definition 25 (cp. [16])
Let X be a compact Hausdorff space and let $X \curvearrowleft G$ be a group action from the right. Then:

1. *Define a equivalence relation on X via*

$$\forall x, y \in X : x \sim y \Leftrightarrow \exists g \in G : x = yg$$

2. *Define the topological space $X/G := X/\sim$ together with the quotient topology.*

We now show, that this is well defined:

Lemma 20 (cp. [16])
Let anything be as in the last definition. Then:

1. *$\sim$ is an equivalence relation on X*

2. *$X/G := X/\sim$ is a compact Hausdorff space.*

Proof. 1.) Reflexivity follows directly since $x = xe$ holds for all $x \in X$. Further symmetry holds, since for $x, y \in X$ with $x = yg$ we have $y = xg^{-1}$. Transitivity follows since for $x, y, z \in X$ with $x = yg$ and $y = zh$ we have $x = zhg$.
2.) Please recall first, that quotient spaces of compact spaces are compact, since the canonical projection $\tilde{\pi} : X \to X/G$ is a continuous surjection and images of compact sets under continuous maps are compact. Further Cor. 3.7.23 of [16] gives that X/G is Hausdorff since X is locally compact as a compact space and G is a compact topological group. □

4.2. Spectral Theory of C^*-Dynamical Systems

We first analyze the prototypical example of the algebra of continuous functions over a compact Hausdorff space. Later we will use this to investigate the general case of abelian, unital C^*-algebras.

Lemma 21
Let X be a compact Hausdorff space and $\tilde{\Phi} : X \curvearrowleft G$ be a group action from the right. Then:

1. *$(C(X), G, \Phi)$ with $\Phi : G \curvearrowright C(X), \Phi(g, f) = f(\cdot g)$ is a C^*-dynamical system.*

2. *$\mathrm{Fix}_G\,(C(X))$ and $C(X/G)$ are isometrically isomorphic as unital C^*-algebras.*

Proof. 1.) That Φ_g is a $*$-morphism for all $g \in G$ can be shown straightforwardly. Further Φ_g is bijective, since $\forall g \in G : \Phi_g^{-1} = \Phi_{g^{-1}}$. That Φ_g is an isometry for all $g \in G$ follows since $\tilde{\Phi}_g$ is surjective for all $g \in G$:

$$\begin{aligned}
\forall f \in C(X) : \sup_{x \in X} |gf(x)| &= \sup_{x \in X} |f(\tilde{\Phi}_g(x))| \\
&= \sup_{y \in \Phi_g(X)} |f(y)| \\
&= \sup_{y \in X} |f(y)|
\end{aligned}$$

Hence it is an isometry for each $g \in G$. Continuity was shown in Lemma 2.5 of [37].
3.) We define the conjectured $*$-isomorphism:

$$\Psi : C(X/G) \to \mathrm{Fix}_G(C(X)), f \mapsto f \circ \pi$$

We now show, that Ψ is a well defined, surjective and isometric $*$-morphism. Then the claim follows, since isometries are injective and continuous.

We first show, that it is a $*$-morphism:

$$\begin{aligned}\Psi(fg) &= (fg) \circ \pi = (f \circ \pi)(g \circ \pi) = \Psi(f)\Psi(g)\\ \Psi(\alpha f + \beta g) &= (\alpha f + \beta g) \circ \pi = \alpha(f \circ \pi) + \beta(g \circ \pi) = \alpha\Psi(f) + \beta\Psi(g)\\ \Psi(\bar{f}) &= \bar{f} \circ \pi = \overline{(f \circ \pi)} = \overline{\Psi(f)}\end{aligned}$$

That it defines an isometry follows by surjectivity of π:

$$\begin{aligned}\sup_{x\in X} |(\Psi f)(x)| &= \sup_{x \in X} |f \circ \pi(x)|\\ &= \sup_{x\in\pi(X)} |f(x)|\\ &= \sup_{x\in X/G} |f(x)|\end{aligned}$$

Now we show, that Ψ is well defined, i.e., that the image of Ψ lies really in $\mathrm{Fix}_G(C(X))$. Therefore let $f \in C(X/G)$ and $g \in G$. Then $\Psi(f) = f \circ \pi \in C(X)$ as a composition of continuous maps. Further:

$$\begin{aligned}\Psi(f)(\cdot g) &= f \circ \pi(\cdot g)\\ &= f \circ \pi(\cdot)\\ &= \Psi(f)\end{aligned}$$

And hence $\Psi(f) \in \mathrm{Fix}_G(C(X))$. Now we show surjectivity. Therefore let $\tilde{f} \in \mathrm{Fix}_G(C(X))$. Then define the following function:

$$f : X/G \to \mathbb{C}, [x] \mapsto \tilde{f}(x)$$

We first show, that f is well defined. Therefore let $y \in [x]$ with $x \neq y$. Then there is a $g \in G$ such that $y = xg$. Then G-invariance of $\tilde{f}$ implies:

$$\tilde{f}(y) = \tilde{f}(xg) = \tilde{f}(x)$$

Hence f is well defined. Further recall from elementary topology, that f is continuous if and only if $f \circ \pi$ is continuous. Now observe:

$$f \circ \pi : X \to \mathbb{C}, x \mapsto \tilde{f}(x)$$

This gives $f \circ \pi = \tilde{f}$ and hence $f \circ \pi$ is continuous since $\tilde{f}$ is. Further this gives $\Psi f = f \circ \pi = \tilde{f}$ and hence we have shown surjectivity. Finally we show, that it is unital. Therefore let $\mathbb{1}$ be the unit on $C(X)$. We then have, that $\mathbb{1} \circ \pi : X/G \to \mathbb{C}, [x] \mapsto 1$ and hence $\Psi(\mathbb{1}) = \mathbb{1}_{X/G}$, where $\mathbb{1}_{X/G}$ is the unit on $C(X/G)$. Hence Ψ is unital. Then the claim follows. □

We now use this prototypical example to show the following theorem:

Theorem 28
Let $(\mathfrak{U}, G, \Phi)$ be a C^*-dynamical system with abelian, unital C^*-algebra $\mathfrak{U}$. Then:

1. $\tilde{\Phi} : \Delta(\mathfrak{U}) \curvearrowleft G, (\chi, g) \mapsto \chi(g\cdot)$ *is continuous group action from the right.*
2. $\Delta(\mathrm{Fix}_G(\mathfrak{U})) = \Delta(\mathfrak{U})/G$.

Proof. 1.) We first show that $\tilde{\Phi}$ is continuous in product topology on $\Delta(\mathfrak{U}) \times G$, where the topology on G is given by the Gel'fand topology. Therefore let $(x_n)_{n\in I}$ be a net in $\Delta(\mathfrak{U}) \times G$ with $\lim x_n = x \in \Delta(\mathfrak{U}) \times G$. Since the canonical projections from $\Delta(\mathfrak{U}) \times G$ on its factors are continuous, we can write $x_n = (\chi_n, g_n)$ with nets $(\chi_n)_{n\in I} \subset \Delta(\mathfrak{U})$ and $(g_n)_{n\in I} \subset G$. Further we have $x = (\chi, g)$ and

$$\begin{aligned}\lim \chi_n &= \chi \\ \lim g_n &= g\end{aligned}$$

We now show $\lim \tilde{\Phi}(\chi_n, g_n) = \tilde{\Phi}(\chi, g)$, since then continuity follows. I.e. we have to show:

$$\forall a \in \mathfrak{U} : \forall \epsilon > 0 : \exists N \in I : \forall n \geq N : |\chi_n(g_n a) - \chi(ga)| < \epsilon$$

Therefore observe:

$$\begin{aligned}|\chi_n(g_n a) - \chi(ga)| &= |\chi_n(g_n a) - \chi_n(ga) + \chi_n(ga) - \chi(ga)| \\ &\leq |\chi_n(g_n a) - \chi_n(ga)| + |\chi_n(ga) - \chi(ga)|\end{aligned}$$

We first consider the second summand. Since $(\chi_n)_{n\in I}$ is a net with $\lim \chi_n = \chi$ in Gel'fand topology, we have:

$$\forall b \in \mathfrak{U} : \forall \epsilon > 0 : \exists N \in I : \forall n \geq N : |\chi_n(b) - \chi(b)| < \frac{\epsilon}{2}$$

End hence especially this holds for $b = ga$.

Now consider the first summand. Therefore observe:

$$\begin{aligned}|\chi_n(g_n a) - \chi_n(ga)| &= |\chi_n(g_n a - ga)| \\ &\leq \sup_{\chi \in \Delta(\mathfrak{U})} |\chi(g_n a - ga)|\end{aligned}$$

Now by theorem 3 we have:

$$\forall a \in \mathfrak{U} : \sup_{\chi \in \Delta(\mathfrak{U})} |\chi(a)| = \|a\|$$

Hence we have:

$$\sup_{\chi \in \Delta(\mathfrak{U})} |\chi(g_n a - ga)| = \|g_n a - ga\|$$

Now by strong continuity of Φ we have:

$$\forall a \in \mathfrak{U} : \forall \epsilon > 0 : \exists M \in I : \forall n \geq M : \|\Phi(g_n, a) - \Phi(g, a)\| < \frac{\epsilon}{2}$$

Hence we have for all $n \geq \max\{N, M\}$:

$$|\chi_n(g_n a) - \chi(ga)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

And hence $\lim \tilde{\Phi}(\chi_n, g_n) = \tilde{\Phi}(\chi, g)$, which gives continuity in product topology.
3.) We split the proof into two assertions:

Assertion 1: Let $\mathfrak{U}_1, \mathfrak{U}_2$ be two C^*-algebras with $\mathfrak{U}_1 \cong \mathfrak{U}_2$. Then $\Delta(\mathfrak{U}_1) = \Delta(\mathfrak{U}_2)$.

Proof of Assertion 2: Let $\Psi : \mathfrak{U}_1 \to \mathfrak{U}_2$ denote the isomorphism between $\mathfrak{U}_1$ and $\mathfrak{U}_2$. Recall from lemma 13, that injectivity of Ψ implies, that the map

$$\Psi^* : \Delta(\mathfrak{U}_2) \to \Delta(\mathfrak{U}_1), \chi_{\mathfrak{U}_2} \mapsto \chi_{\mathfrak{U}_2} \circ \Psi$$

is surjective and continuous. Hence it remains to show, that surjectivity of Ψ implies, that Ψ^* is injective. That this is indeed the case follows easily. Let $\chi_1, \chi_2 \in \Delta(\mathfrak{U}_2)$ with $\chi_1 \neq \chi_2$. This means, that there is an $a \in \mathfrak{U}_2$ with $\chi_1(a) \neq \chi_2(a)$. By surjectivity of Ψ there is a $b \in \mathfrak{U}_1$ with $\Psi(b) = a$. Now we have:

$$\Psi^*\chi_1(b) = \chi_1 \circ \Psi(b) = \chi_1(a) \neq \chi_2(a) = \chi_2 \circ \Psi(b) = \Psi^*\chi_2(b)$$

Hence $\Psi^*\chi_1 \neq \Psi^*\chi_2$ holds and hence Ψ^* is injective. Now the claim follows, since $\Delta(\mathfrak{U}_1)$ and $\Delta(\mathfrak{U}_2)$ are compact Hausdorff spaces and hence a continuous bijection between them automatically is a homeomorphism.

Assertion 2: Let $\mathcal{G} : \mathfrak{U} \to C(\Delta(\mathfrak{U}))$ be the Gel'fand transform. Then:

$$\mathcal{G}|_{\mathrm{Fix}_G(\mathfrak{U})} : \mathrm{Fix}_G(\mathfrak{U}) \to \mathrm{Fix}_G(C(\Delta(\mathfrak{U})))$$

is an isometric isomorphism of C^*-algebras.

Proof of Assertion 2: We have to show, that $\mathcal{G}|_{\mathrm{Fix}_G(\mathfrak{U})} : \mathrm{Fix}_G(\mathfrak{U}) \to \mathrm{Fix}_G(C(\Delta(\mathfrak{U})))$ is a unital $*$-morphism, an isometry, well-defined and surjective. Then the claim follows, since isometries are automatically injective and continuous. Observe first, that since $\mathcal{G}$ is an isometric, unital $*$-morphism, so is $\mathcal{G}|_{\mathrm{Fix}_G(\mathfrak{U})}$. We now show surjectivity. Therefore let $\tilde{f} \in \mathrm{Fix}_G(C(\Delta(\mathfrak{U})))$, i.e. $\tilde{f} \in C(\Delta(\mathfrak{U}))$ with $\tilde{f}(\cdot g) = \tilde{f}$ for all $g \in G$. By surjectivity of $\mathcal{G}$, there is an $a \in \mathfrak{U}$ such that $\forall \chi \in \Delta(\mathfrak{U}) : \tilde{f}(\chi) = \chi(a)$. Observe, that in this concrete situation, the G-invariance of $\tilde{f}$ implies:

$$\forall \chi \in \Delta(\mathfrak{U}) \forall g \in G : \chi(ga) = \chi(a)$$

We now have to show, that $(\forall \chi \in \Delta(\mathfrak{U}) : \chi(a) = \chi(b)) \Rightarrow a = b$. Therefore observe, that $\chi(a) = \chi(b) \Leftrightarrow |\chi(a) - \chi(b)| = 0$. Further we have $|\chi(a) - \chi(b)| = |\chi(a - b)|$. Now we have:

$$\forall \chi \in \Delta(\mathfrak{U}) : |\chi(a - b)| = 0 \Rightarrow \sup_{\chi \in \Delta(\mathfrak{U})} |\chi(a - b)| = 0$$

And hence we have $\|a - b\| = 0$ by theorem 3. Since norms are positive-definite, this gives $a - b = 0$. Hence $\mathcal{G}|_{\mathrm{Fix}_G(\mathfrak{U})}$ is surjective. Finally we show well definedness. Therefore let $a \in \mathrm{Fix}_G(\mathfrak{U})$. Then:

$$\begin{aligned} \forall \chi \in \Delta(\mathfrak{U}) : \check{a}(\chi g) &= (\chi g)(a) \\ &= \chi(ga) \\ &= \chi(a) \\ &= \check{a}(\chi) \end{aligned}$$

And hence $\check{a} \in \mathrm{Fix}_G(C(\Delta(\mathfrak{U})))$.

Now the claim follows since by above and by lemma 21 it holds, that:

$$\begin{aligned}\Delta(\mathrm{Fix}_G(\mathfrak{U})) &= \Delta(\mathcal{G}(\mathrm{Fix}_G(\mathfrak{U}))) \\ &= \Delta(\mathrm{Fix}_G((C(\Delta(\mathfrak{U})))) \\ &= \Delta(C(\Delta(\mathfrak{U})/G))\end{aligned}$$

And by lemma 3 we finally have:

$$\Delta(C(\Delta(\mathfrak{U})/G)) = \Delta(\mathfrak{U})/G$$

□

4.3. Projective Limits of Group Actions on Compact Hausdorff Spaces

In the last section we have investigated, how the spectrum of the fixed point subalgebra of a C^*-dynamical system is related to the spectrum of the full C^*-algebra. In the next three sections we want to investigate, to which extent this knowledge gives us information on the spectral theory of the corresponding inductive limit C^*-dynamical system. Therefore in this section first the topological side is investigated by considering projective families of group actions on projective families of compact Hausdorff spaces. Therefore we first define, what is meant by a projective family of group actions:

Definition 26 (cp. p. 177 of [31])
Let G be a compact, topological Hausdorff group, L be a label set and $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ be an projective family of compact Hausdorff spaces. Then: A family of group actions $\tilde{\Phi}_\gamma : X_\gamma \curvearrowleft G$ is called consistent, iff:

$$\forall \gamma' \geq \gamma : \forall x_{\gamma'} \in X_{\gamma'} : p_{\gamma\gamma'}(x_{\gamma'} g) = p_{\gamma\gamma'}(x_{\gamma'}) g$$

I.e. iff $p_{\gamma\gamma'}$ is G-equivariant with respect to those group actions for each $\gamma \leq \gamma' \in L$.

Further the projective limit of such a consistent family of group actions is a group action $\lim_{\leftarrow} \tilde{\Phi}_\gamma : \lim_{\leftarrow} X_\gamma \curvearrowleft G$ with:

$$\forall \gamma \in L : \forall x \in \lim_{\leftarrow} X_\gamma : p_\gamma(xg) = p_\gamma(x) g$$

In the following a consistent family of group actions corresponding to a projective family of topological spaces will also be called a projective family of group actions thereon.

We now want to investigate, if the projective limit of group actions exists and if it is unique:

Theorem 29 (cp. p. 177 of [31])
Let G be a compact, topological Hausdorff group, L be a label set and $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ be a projective family of compact Hausdorff spaces. Let further $\tilde{\Phi}_\gamma : X_\gamma \curvearrowleft G$ be a consistent family of group actions as defined in above definition. Then: The projective limit of $\left(\tilde{\Phi}_\gamma : X_\gamma \curvearrowleft G\right)_{\gamma\in L}$ exists and is unique.

Proof. We first define the conjectured projective limit group action:

$$\tilde{\Phi} : \lim_{\leftarrow} X_\gamma \curvearrowleft G, ((x_\gamma)_{\gamma \in L}, g) \mapsto (x_\gamma g)_{\gamma \in L}$$

We now want to show, that this defines the unique projective limit group actions. Therefore we show the following Assertions:

Assertion 1 : Let $g \in G$ and $(x_\gamma)_{\gamma \in L} \in \lim_{\leftarrow} X_\gamma$. Then $\tilde{\Phi}(g, (x_\gamma)_{\gamma \in L}) \in \lim_{\leftarrow} X_\gamma$.

Proof of Assertion 1: Let $g \in G$ and $(x_\gamma)_{\gamma \in L} \in \lim_{\leftarrow} X_\gamma$. Let further $\gamma' \geq \gamma \in L$. Then:

$$\begin{aligned} p_{\gamma\gamma'}\Big(\pi_\gamma \circ \tilde{\Phi}\left((x_\gamma)_{\gamma \in L}, g\right)\Big) &= p_{\gamma\gamma'}(x_{\gamma'} g) \\ &= p_{\gamma\gamma'}(x_{\gamma'}) g \\ &= x_\gamma g \\ &= \pi_\gamma \circ \Big(\tilde{\Phi}((x_\gamma)_{\gamma \in L}, g)\Big) \end{aligned}$$

Where $\pi_\gamma : \prod_{\gamma \in L} X_\gamma \to X_\gamma$ is the canonical projection.
And hence Assertion 1 follows.

Assertion 2: $\tilde{\Phi}$ is continuous in product topology.

Proof of Assertion 2: Please observe first, that $\tilde{\Phi}$ is continuous in product topology iff $\forall \gamma \in L : p_\gamma \circ \tilde{\Phi} : \lim_{\leftarrow} X_\gamma \times G \to X_\gamma((x_\gamma)_{\gamma \in L}, g) \mapsto x_\gamma g$ is continuous in product topology. To show the latter, let $(\zeta_n)_{n \in I} \subset \lim_{\leftarrow} X_\gamma \times G$ be a net with $\lim \zeta_n = \zeta \in \lim_{\leftarrow} \times G$. By continuity of the canonical projections from $\lim_{\leftarrow} X \times G$ on its factors we can write $\zeta_n = (x_n, g_n) \in \lim_{\leftarrow} X_\gamma \times G$ and $\zeta = (x, g) \in \lim_{\leftarrow} X_\gamma \times G$ such that $\lim x_n = x$ and $\lim g_n = g$. Now $\lim x_n = x$ means, that for each $\gamma \in L$ the net $(p_\gamma x_n)_{n \in I}$ converges towards $p_\gamma(x)$. Hence by continuity of $\tilde{\Phi}_\gamma$ we have, that $\lim \tilde{\Phi}_\gamma(p_\gamma x_n, g_n) = \tilde{\Phi}_\gamma(p_\gamma x, g)$. Hence the assertion follows.

Assertion 3: $\tilde{\Phi}_g(\tilde{\Phi}_h(\cdot)) = \tilde{\Phi}_{hg}(\cdot)$ and $\tilde{\Phi}_g^{-1}(\cdot) = \tilde{\Phi}_{g^{-1}}(\cdot)$.

Proof of Assertion 3: Since those properties hold for $p_\gamma \circ \tilde{\Phi}_g = \tilde{\Phi}_\gamma(\cdot, g)$ for all $\gamma \in L$ it follows directly, that they hold for $\tilde{\Phi}_g$.

Assertion 4: $\tilde{\Phi}$ is really the projective limit group action.

Proof of Assertion 4: This follows directly, since by definition $p_\gamma \circ \tilde{\Phi}_g = \tilde{\Phi}_\gamma(\cdot, g)$ holds directly.

Assertion 5: $\tilde{\Phi}$ is the unique projective limit group action.

Proof of Assertion 5: Assume, that there exists a $\tilde{\Psi} : \lim_{\leftarrow} X_\gamma \curvearrowleft G$ such that $\forall g \in G :$ $p_\gamma \circ \tilde{\Psi}_g = \tilde{\Phi}_\gamma(\cdot, g)$ holds, but $\tilde{\Psi} \neq \tilde{\Phi}$. Observe that the latter means, that there is a $g \in G$ and a $(x_\gamma)_{\gamma \in L} \in \lim_{\leftarrow} X_\gamma$ with:

$$\tilde{\Psi}\left((x_\gamma)_{\gamma \in L}, g\right) \neq \tilde{\Phi}\left((x_\gamma)_{\gamma \in L}, g\right)$$

But this, means, that there is a $\gamma' \in L$ such that

$$\tilde{\Phi}_{\gamma'}(x_{\gamma'}, g) = p_{\gamma'} \circ \tilde{\Psi}\left((x_\gamma)_{\gamma \in L}, g\right) \neq p_{\gamma'} \circ \tilde{\Phi}\left((x_\gamma)_{\gamma \in L}, g\right) = \tilde{\Phi}_{\gamma'}(x_{\gamma'}, g)$$

which is a contradiction. Hence $\tilde{\Phi}$ is the unique projective limit group action. □

We have now understood, how group actions on a projective family define a group action on the projective limit. We now want to understand, how the corresponding quotient spaces are related. But before doing so we need three lemmata:

Lemma 22
Let X_1, X_2 be compact Hausdorff spaces and $\tilde{\Phi}_1 : X_1 \curvearrowleft G$, $\tilde{\Phi}_2 : X_2 \curvearrowleft G$ be two G-actions. Let $f : X_1 \to X_2$ be continuous and G-equivariant, i.e.:

$$\forall g \in G : f(\cdot g) = f(\cdot) g$$

Then: The map

$$\tilde{f} : X_1/G \to X_2/G, [x] \mapsto [f(x)]$$

is well defined and continuous.

Proof. We first show, that $\tilde{f}$ as defined above is well defined. Therefore let $x, y \in [x] \in X_1/G$, i.e. $\exists g \in G : x = yg$. Then:

$$[f(x)] = [f(yg)] = [f(y)g] = [f(y)]$$

And hence $\tilde{f}$ is well defined. Now towards continuity: Therefore let $U \subset X_2/G$ be open. Now let $\pi_i : X \to X_i/G$ denote the corresponding quotient maps. By continuity of π_2 we have, that $\pi_2^{-1}(U)$ is open. By continuity of f we have further, that $f^{-1}\left(\pi_2^{-1}(U)\right)$ is open. Now we have (cp. Lem. 3.7.11 of [16]), that the π_i are open maps in this case. Hence $\pi_1\left(f^{-1}\left(\pi_2^{-1}(U)\right)\right)$ is open. Now $\tilde{f} = \pi_2 \circ f \circ \pi_1^{-1}$, and hence $\tilde{f}^{-1}(U)$ is open. Hence $\tilde{f}$ is continuous. □

Lemma 23
Let X, Y be compact Hausdorff spaces, $X \curvearrowleft G$ be a group action, $\tilde{\pi} : X \to Y$ be a surjection and further $\pi : X \to X/G$ be the canonical projection. Let further

$$\forall x, y \in X : \pi(x) \neq \pi(y) \Leftrightarrow \tilde{\pi}(y) \neq \tilde{\pi}(y) \tag{4.1}$$

Then: $X/G \cong Y$.

Proof. We define the conjectured homeomorphism as:

$$\Psi : Y \to X/G, y \mapsto \pi\left(\tilde{\pi}^{-1}(y)\right)$$

We now have to show, that this defines a function, i.e. that it is left-total and right-unique. Further we have to show, that it is injective, surjective and continuous. Since a bijective continuous map between a compact space and a Hausdorff space is automatically a homeomorphism, then the claim follows.

We first show, that Ψ is left-total. Therefore let $y \in Y$. By surjectivity of $\tilde{\pi}$, there is a $x \in X$ s.th. $\tilde{\pi}(x) = y$. Since π is a function, it then follows, that $\Psi(\{y\}) \neq \emptyset$.

We now show, that Ψ is right-unique. Therefore let $y \in Y$. Now let $y_1, y_2 \in \tilde{\pi}^{-1}(y)$ with $y_1 \neq y_2$. We then have $\tilde{\pi}(y_1) = \tilde{\pi}(y_2)$. By eq. (4.1) this implies $\pi(y_1) = \pi(y_2)$. Hence $y \mapsto \pi(\tilde{\pi}^{-1}(y))$ is right-unique.

Hence we have shown, that Ψ is well defined. We now show injectivity. Therefore let $y_1 \neq y_2 \in Y$. Then $\tilde{\pi}^{-1}(y_1) \cap \tilde{\pi}^{-1}(y_2) = \emptyset$. Now let $\tilde{y}_1 \in \tilde{\pi}^{-1}(y_1)$ and $\tilde{y}_2 \in \tilde{\pi}^{-1}(y_2)$, i.e. $\tilde{y}_1 \neq \tilde{y}_2$. Further we have $\tilde{\pi}(\tilde{y}_1) \neq \tilde{\pi}(\tilde{y}_2)$. Then by eq. (4.1) this implies $\pi(\tilde{y}_1) \neq \pi(\tilde{y}_2)$ and injectivity follows.

Now we show surjectivity. Therefore let $x \in X/G$. Since π is surjective, there is a $\tilde{x} \in X$ with $\pi(\tilde{x}) = x$. Then $\Psi(\tilde{\pi}(\tilde{x})) = x$ and hence surjectivity follows.

We now show continuity of Ψ^{-1}. Therefore please note, that the inverse of Ψ is given by $\Psi^{-1} = \tilde{\pi} \circ \pi^{-1}$. Now observe further, that and $\tilde{\pi}$ is continuous and further π^{-1} is continuous since π is open in our case (cp. Lem. 3.7.11 of [16]). Hence Ψ^{-1} is continuous as a composition of continuous maps and hence Ψ is a homeomorphism. □

The following lemma is a well-known result from point-set topology, therefore we won't show it.

Lemma 24 (cp. Prop. 5.1 of [12])
Let X be a topological space. Then the following are equivalent:

1. *X is compact.*
2. *Every infinite collection of closed subsets $(X_i \subset X)_{i \in I}$ which satisfies the finite intersection property*
$$\forall S \subset I \text{ finite } : \bigcap_{i \in S} X_i \neq \emptyset$$
has non-empty total intersection:
$$\bigcap_{i \in I} X_i \neq \emptyset$$

We now show the final theorem of this section, which gives, that the projective limit of quotient spaces is the quotient of the projective limit:

Theorem 30 (cp. [31] for a different proof)
Let $(X_\gamma, p_{\gamma\gamma'})_{\gamma \leq \gamma' \in L}$ be a projective family of compact Hausdorff spaces and let $\tilde{\Phi}_\gamma : X_\gamma \curvearrowleft G$ be a projective family of Group-actions.
Then:

1. *$(X_\gamma/G, \tilde{p}_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ with*
$$\tilde{p}_{\gamma\gamma'} : X_{\gamma'}/G \to X_\gamma/G, [x_{\gamma'}] \mapsto [p_{\gamma\gamma'} x_{\gamma'}]$$
is a projective family of compact Hausdorff spaces.
2. $\lim_{\leftarrow} (X_\gamma/G) = (\lim_{\leftarrow} X_\gamma)/G$

Proof. 1.) Therefore we have to show, that X_γ/G is a compact Hausdorff space, that $\tilde{p}_{\gamma\gamma'}$ is well defined, that $\tilde{p}_{\gamma\gamma'}$ satisfies the composition properties demanded in the definition of a projective system and that $\tilde{p}_{\gamma\gamma'}$ is continuous.

Assertion 1: For each $\gamma \in L$ it holds, that X_γ/G is a compact Hausdorff space.

Proof of Assertion 1: This was already shown in lemma 20.

Assertion 2: $\tilde{p}_{\gamma\gamma'}$ is well defined and continuous.

Proof of Assertion 2: Follows with lemma 22 and definition 26.

Assertion 3: $\tilde{p}_{\gamma\gamma} = id.$ and $\tilde{p}_{\gamma\gamma'} \circ \tilde{p}_{\gamma'\gamma''} = \tilde{p}_{\gamma\gamma''}$.

Proof of Assertion 3: This follows directly by well definedness of $\tilde{p}_{\gamma\gamma'}$ and by the fact, that $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma' \in L}$ is an projective system.

With this the claim follows.

2.) We use lemma 23 to show this. Therefore we first construct the following map:

$$\tilde{\pi} : \lim_{\leftarrow} X_\gamma \to \lim_{\leftarrow} \left(X_\gamma/G\right), (x_\gamma)_{\gamma \in L} \mapsto ([x_\gamma])_{\gamma \in L}$$

We now show, that this map satisfies the prerequisits of lemma 23.

Lemma for Assertion 1: Let $([y_\gamma])_{\gamma \in L} \in \lim_{\leftarrow}(X_\gamma/G)$. Then $\exists (x_\gamma)_{\gamma \in L} \in \lim_{\leftarrow} X_\gamma$ such that $\forall \gamma \in L : x_\gamma \in [y_\gamma]$.

Proof of Lemma for Assertion 1: Set

$$R := \{(\gamma, \gamma') \in L \times L | \gamma' \geq \gamma\}$$

and set further for all $(\gamma, \gamma') \in R$:

$$C(\gamma, \gamma') := \left\{ (x_\gamma)_{\gamma \in L} \in \prod_{\gamma \in L} X_\gamma \middle| \forall \gamma \in L : x_\gamma \in [y_\gamma] \wedge p_{\gamma\gamma'} x_{\gamma'} = x_\gamma \right\}$$

We split the proof of this "Lemma" into four claims:

Claim 1: $\bigcap_{(\gamma,\gamma') \in R} C(\gamma, \gamma') \subseteq \lim_{\leftarrow} X_\gamma$.

Proof of Claim 1: The claim follows directly in the trivial case $\bigcap_{(\gamma,\gamma') \in R} C(\gamma, \gamma') = \emptyset$. Now consider the non-trivial case of $\bigcap_{(\gamma,\gamma') \in R} C(\gamma, \gamma') \neq \emptyset$. Therefore observe:

$$\bigcap_{(\gamma,\gamma') \in R} C(\gamma, \gamma') = \left\{ (x_\gamma)_{\gamma \in L} \in \prod_{\gamma \in L} \middle| \forall \gamma \in L : x_\gamma \in [y_\gamma] \wedge \forall \gamma' \geq \gamma : p_{\gamma\gamma'} x_{\gamma'} = x_\gamma \right\}$$

And hence the claim follows.

Claim 2: For $(x_\gamma)_{\gamma \in L} \in \bigcap_{(\gamma,\gamma') \in R} C(\gamma, \gamma')$ we have $\forall \gamma \in L : x_\gamma \in [y_\gamma]$.

Proof of Claim 2: This follows directly by the definition of $C(\gamma, \gamma')$.

Claim 3: $C(\gamma, \gamma')$ is closed.

Proof of Claim 3: Therefore observe first, that:

$$C(\gamma,\gamma') = \left\{ (x_\gamma)_{\gamma\in L} \in \prod_{\gamma\in L} X_\gamma \middle| p_{\gamma\gamma'}x_{\gamma'} = x_\gamma \right\} \cap \prod_{\gamma\in L}[y_\gamma]$$

That $\left\{ (x_\gamma)_{\gamma\in L} \in \prod_{\gamma\in L} X_\gamma \middle| p_{\gamma\gamma'}x_{\gamma'} = x_\gamma \right\}$ is closed was shown in Assertion 2 of the second part of theorem 12. We now show, that $\prod_{\gamma\in L}[y_\gamma]$ is closed. Therefore it suffices to show, that the orbits $[y_\gamma]$ are closed, since arbitrary products of closed sets are closed. Therefore define the map

$$\tilde{\Phi}_\gamma(y_\gamma) : G \to X_\gamma, g \mapsto \tilde{\Phi}(y_\gamma, g)$$

and observe, that by definition $[y_\gamma] = \operatorname{im}(\tilde{\Phi}_\gamma)$ holds. Observe further, that $\tilde{\Phi}_\gamma(y_\gamma)$ is continuous, since $\tilde{\Phi}$ is continuous in product topology. Finally $\operatorname{im}(\tilde{\Phi}_\gamma)$ is compact, since G is compact and hence $[y_\gamma] = \operatorname{im}(\tilde{\Phi}_\gamma)$ is closed as a compact subspace of a Hausdorff space. Hence $C(\gamma,\gamma')$ is closed as an intersection of two closed sets.

Claim 4: Let $((\gamma_i,\gamma_i'))_{i\in I} \subset R$ be finite. Then:

$$\bigcap_{i\in I} C(\gamma_i,\gamma_i') \neq \emptyset$$

I.e. $\{C(\gamma,\gamma') \subset \lim_{\leftarrow} X_\gamma | (\gamma,\gamma') \in R\}$ satisfies the finite intersection property.

Proof of Claim 4: Set $\hat{\gamma} \in L$ with $\hat{\gamma} \geq \gamma_i, \gamma_i'$ for all $i \in I$, which exists, since L is partially ordered, directed set. Choose any $x_{\hat{\gamma}} \in [y_{\hat{\gamma}}]$ and define an $(x_\gamma)_{\gamma\in L} \in \prod_{\gamma\in L} X_\gamma$ as follows: Set $x_\gamma = p_{\gamma\hat{\gamma}}x_{\hat{\gamma}}$ for all $\gamma \leq \hat{\gamma}$ and choose for any other $\gamma \in L$ just any $x_\gamma \in [y_\gamma]$. We then have

$$(x_\gamma)_{\gamma\in L} \in \bigcap_{i\in I} C(\gamma_i,\gamma_i')$$

since

$$\forall i \in I : p_{\gamma_i\gamma_i'}x_{\gamma_i} = x_{\gamma_i}$$
$$\forall \gamma \in L : x_\gamma \in [y_\gamma]$$

Now it follows, that

$$\emptyset \neq \bigcap_{(\gamma,\gamma')\in R} C(\gamma,\gamma') \neq 0$$

by lemma 24. Hence the Lemma for Assertion 1 is proven.

Assertion 1: $\tilde{\pi}$ is surjective.

Proof of Assertion 1: Let $([y_\gamma])_{\gamma\in L} \in \lim_{\leftarrow}(X_\gamma/G)$. Then by above Lemma we have, that there is a $(x_\gamma)_{\gamma\in L} \in \lim_{\rightarrow} X_\gamma$ such that $\forall\gamma \in L : x_\gamma \in [y_\gamma]$. Hence $\tilde{\pi}((x_\gamma)_{\gamma\in L}) = ([y_\gamma])_{\gamma\in L}$ and the assertion follows.

Assertion 2: $\tilde{\pi}$ is continuous.

Proof of Assertion 2: Recall: $\tilde{\pi} : \lim_{\leftarrow} X_\gamma \to \lim_{\leftarrow}(X_\gamma/G)$ is continuous if and only if for all $\gamma \in L$ it holds, that $\tilde{p}_\gamma \circ \tilde{\pi} : \lim_{\leftarrow} X_\gamma \to X_\gamma/G$ is continuous. Now we have:

$$\tilde{p}_\gamma \circ \tilde{\pi} : (x_\gamma)_{\gamma \in L} \mapsto [x_\gamma]$$

i.e. $\tilde{p}_\gamma \circ \tilde{\pi} = \pi_\gamma \circ p_\gamma$, where $\pi_\gamma : X_\gamma \to X_\gamma/G$ denotes the quotient map. Since the latter is continuous as a composition of continuous functions, the assertion follows.

Assertion 3: Let $x, y \in \lim_{\leftarrow} X_\gamma$. Then $[(x_\gamma)]_{\gamma \in L}] = [(y_\gamma)_{\gamma \in L}] \in (\lim_{\leftarrow} X_\gamma)/G$ if and only if $([x_\gamma])_{\gamma \in L} = ([y_\gamma])_{\gamma \in L} \in \lim_{\leftarrow}(X_\gamma/G)$.

Proof of Assertion 3: Please observe first the following equivalences:

$$[(x_\gamma)]_{\gamma \in L}] = [(y_\gamma)_{\gamma \in L}] \in \left(\lim_{\leftarrow} X_\gamma\right)/G \Leftrightarrow \exists g \in G : \forall \gamma \in L x_\gamma = y_\gamma g \tag{4.2}$$

$$([x_\gamma])_{\gamma \in L} = ([y_\gamma])_{\gamma \in L} \in \lim_{\leftarrow}(X_\gamma/G) \Leftrightarrow \forall \gamma \in L : \exists g_\gamma \in G : x_\gamma = y_\gamma g_\gamma \tag{4.3}$$

That eq. (4.2) implies eq. (4.3) follows trivially. We now want to show, that eq. (4.3) together with the equivariance of the projection maps implies the left hand side.

Therefore we define first the following set:

$$\forall \gamma \in L : G_\gamma := \{g \in G | x_\gamma = y_\gamma g\}$$

Claim 1 for Assertion 3: For each $\gamma \in L$ it holds, that G_γ is closed.

Proof of Claim 1: Observe first the following alternative description of G_γ:

$$G_\gamma = \{g \in G | \tilde{\Phi}_\gamma(y_\gamma, g) = x_\gamma\}$$

We now define a map

$$\tilde{\Phi}_\gamma(y_\gamma) : G \to X_\gamma, g \mapsto \tilde{\Phi}_\gamma(y_\gamma, g)$$

Please observe that this map is continuous, since $\tilde{\Phi}_\gamma$ is continuous in product topology. With this we can write:

$$G_\gamma = \tilde{\Phi}_\gamma(y_\gamma)^{-1}(\{x_\gamma\})$$

Further observe that X_γ is a Hausdorff space, and hence single point sets are closed. Further $\tilde{\Phi}_\gamma(y_\gamma)$ is continuous, since $\tilde{\Phi}$ is continuous in product topology. Hence G_γ is a continuous preimage of a closed set and hence closed.

Claim 2 for Assertion 3: The family of closed sets $(G_\gamma \subset G)_{\gamma \in L}$ satisfies the finite intersection property.

Proof of Claim 2: Let $S \subset L$ be finite. Then there is a $\hat{\gamma} \in L$ such that $\forall \gamma \in S : \gamma \leq \hat{\gamma}$, since L is a directed poset. Further by eq. (4.2) we have, that $G_\gamma \neq \emptyset$ for each $\gamma \in L$ and especially $G_{\hat{\gamma}} \neq \emptyset$. Now let $g_{\hat{\gamma}} \in G_{\hat{\gamma}}$ and let $\gamma \in S$. We then have:

$$x_\gamma = p_{\gamma\hat{\gamma}} x_{\hat{\gamma}}$$
$$y_\gamma = p_{\gamma\hat{\gamma}} y_{\hat{\gamma}}$$

Further we have $x_{\hat{\gamma}} = y_{\hat{\gamma}} g_{\hat{\gamma}}$. By equivariance of the projection maps we now have:

$$x_\gamma = p_{\gamma\hat{\gamma}} x_{\hat{\gamma}} = p_{\gamma\hat{\gamma}}(y_{\hat{\gamma}} g_{\hat{\gamma}}) = (p_{\gamma\hat{\gamma}} y_{\hat{\gamma}})\, g_{\hat{\gamma}} = y_\gamma g_{\hat{\gamma}}$$

I.e. we have shown, that $\forall\gamma \in S : g_{\hat{\gamma}} \in G_\gamma$ and hence we have shown $G_{\hat{\gamma}} \subset G_\gamma$ for all $\gamma \in S$. Hence:

$$\emptyset \neq G_{\hat{\gamma}} \subset \bigcap_{\gamma\in S} G_\gamma$$

And hence $(G_\gamma \subset G)_{\gamma\in L}$ satisfies the finite intersection property.

With lemma 24 it now follows, that

$$\bigcap_{\gamma\in L} G_\gamma \neq \emptyset$$

since G is compact Hausdorff space. Now

$$\bigcap_{\gamma\in L} G_\gamma = \{g \in G | \forall\gamma \in L : x_\gamma = y_\gamma g\}$$

Hence we have found $g \in G$ such that $\forall\gamma \in L : x_\gamma = y_\gamma g$. Hence Assertion 3 is shown. □

4.4. Inductive Limits of C^*-Dynamical Systems

We first define the following:

Definition 27
The family $\left((\mathfrak{U}_\gamma, G, \Phi_\gamma), \phi_{\gamma'\gamma}\right)_{\gamma,\gamma'\in L}$ is called an inductive family of C^*-dynamical systems iff:

1. *$(\mathfrak{U}_\gamma, G, \Phi_\gamma)$ is a C^*-dynamical system for each $\gamma \in L$.*
2. *$(\mathfrak{U}_\gamma, \phi_{\gamma'\gamma})_{\gamma,\gamma'\in L}$ is an inductive family of C^*-algebras.*
3. *$\forall\gamma' \geq \gamma : \forall a_\gamma \in \mathfrak{U}_\gamma : \phi_{\gamma'\gamma}(g a_\gamma) = g\phi_{\gamma'\gamma}(a_\gamma)$.*

In this case we call the $(\Phi_\gamma : G \curvearrowright \mathfrak{U}_\gamma)_{\gamma\in L}$ an inductive family of C^*-group actions. Further the inductive limit of above inductive family of C^*-dynamical systems is an C^*-dynamical system

$$\left(\lim_{\rightarrow} \mathfrak{U}_\gamma, G, \lim_{\rightarrow} \Phi_\gamma\right)$$

such that:

$$\forall\gamma \in L : \forall a_\gamma \in \mathfrak{U}_\gamma : \phi_\gamma(g a_\gamma) = g\phi_\gamma(a_\gamma)$$

In this case we call $\lim_\rightarrow \Phi_\gamma : G \curvearrowright \lim_\rightarrow X_\gamma$ the inductive limit C^*-group action.

With this we show the following theorem:

Theorem 31
Let $\left((\mathfrak{U}_\gamma, G, \Phi_\gamma), \phi_{\gamma'\gamma}\right)_{\gamma,\gamma'\in L}$ be an inductive family of C^*-dynamical systems.
Then: Its inductive limit exists and is unique.

Proof. We define the conjectured inductive limit C^*-group action on a dense subset of $\lim_\rightarrow \mathfrak{U}_\gamma$:

$$\forall g \in G : \Phi_g : \bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma) \to \bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma), \phi_\gamma(a_\gamma) \mapsto \phi_\gamma(g a_\gamma)$$

We now want to show, that the unique extension of this map, given by lemma 9, exists and satisfies the properties of an inductive limit C^*-group action.

Assertion 1: Φ_g defined as above is well defined.

Proof of Assertion 1: This follows by the usual discussion. Let $\phi_\gamma(a_\gamma) = \phi_{\gamma'}(a_{\gamma'})$. Then there is a $\hat{\gamma} \geq \gamma, \gamma'$. We then have: $\phi_\gamma(a_\gamma) = \phi_{\hat{\gamma}}(a_{\hat{\gamma}}) = \phi_{\gamma'}(a_{\gamma'})$ for $a_{\hat{\gamma}} = \phi_{\hat{\gamma}\gamma}(a_\gamma) = \phi_{\hat{\gamma}\gamma'}(a_{\gamma'})$. We then have:

$$\phi_{\hat{\gamma}}(a_{\hat{\gamma}}) = \phi_{\hat{\gamma}}(g\phi_{\hat{\gamma}\gamma}a_\gamma) = \phi_\gamma(ga_\gamma)$$
$$\phi_{\hat{\gamma}}(a_{\hat{\gamma}}) = \phi_{\hat{\gamma}}(g\phi_{\hat{\gamma}\gamma'}a_{\gamma'}) = \phi_{\gamma'}(ga_{\gamma'})$$

and hence it is well-defined.

Assertion 2: Φ_g satisfies the prerequisits of lemma 10, i.e. for each $g \in G$ it holds, that Φ_g is an isometric $*$-isomorphism.

Proof of Assertion 2: We now have to show, that for each $g \in G$ it holds, that

$$\Phi_g : \bigcup_{\gamma \in L} \operatorname{im}(\phi_\gamma) \to \bigcup_{\gamma \in L} \operatorname{im}(\phi_\gamma), \phi_\gamma(a_\gamma) \mapsto \phi_\gamma(ga_\gamma)$$

is an isometric $*$-isomorphism. We first show, that it is a $*$-morphism. Therefore recall, that for each $g \in G$ we have, that

$$\Phi_\gamma(g, \cdot) : \mathfrak{U}_\gamma \to \mathfrak{U}_\gamma$$

is a $*$-morphism. Now let without loss of generality $a, b \in \operatorname{im}(\phi_\gamma)$ and $\alpha, \beta \in \mathbb{C}$. Then:

$$\begin{aligned}
\Phi_g(\alpha a + \beta b) = \phi_\gamma(g(\alpha a_\gamma + \beta b_\gamma)) = \alpha\phi_\gamma(ga) + \beta\phi_\gamma(gb) = \alpha\Phi_g(a) + \beta\Phi_g(b) \\
\Phi_g(ab) = \phi_\gamma(ga_\gamma b_\gamma) = \phi_\gamma((ga_\gamma)(gb_\gamma)) = \phi_\gamma(ga_\gamma)\phi_\gamma(gb_\gamma) = \Phi_g(a)\Phi_g(b) \\
\Phi_g(a^*) = \phi_\gamma(ga_\gamma^*) = \phi_\gamma((ga_\gamma)^*) = \phi_\gamma(ga_\gamma)^* = \Phi_g(a_\gamma)^*
\end{aligned}$$

and hence it is a $*$-morphism.

We now show, that Φ_g defines an isometry and hence also injective. Therefore let $a \in \operatorname{im}(\phi_\gamma)$. Then observe:

$$\begin{aligned}
\|\Phi_g(a)\| &= \|\phi_\gamma(ga_\gamma)\| \\
&= \|\Phi_\gamma(g, a_\gamma)\|_\gamma \\
&= \|a_\gamma\|_\gamma
\end{aligned}$$

Where we have used, that $\Phi_\gamma(g, \cdot)$ and ϕ_γ are isometries.

Now surjectivity. Let $b \in \operatorname{im}(\phi_\gamma)$ with $b = \phi_\gamma(b_\gamma)$. We set $a_\gamma = g^{-1}b_\gamma$. With this we have:

$$g\phi_\gamma(g^{-1}a_\gamma) = \phi_\gamma(b_\gamma) = b$$

and hence surjectivity is shown. The claim then follows by lemma 10

Assertion 3: $\Phi : G \to \operatorname{Aut}(\lim_\to \mathfrak{U}_\gamma)$ is a group homomorphism.

Proof of Assertion 3: Thise follows directly, since on the dense subset $\bigcup_{\gamma\in L} \mathrm{im}\phi_\gamma$ the following holds:

$$\Phi_g \circ \Phi_h|_{\bigcup_{\gamma\in L} \mathrm{im}\phi_\gamma} = \Phi_{gh}|_{\bigcup_{\gamma\in L} \mathrm{im}\phi_\gamma}$$
$$\Phi_g^{-1}|_{\bigcup_{\gamma\in L} \mathrm{im}\phi_\gamma} = \Phi_{g^{-1}}|_{\bigcup_{\gamma\in L} \mathrm{im}\phi_\gamma}$$

Now the assertion follows by uniqueness of the extension given by lemma 10.

Assertion 4: $\Phi : G \to \mathrm{Aut}(\lim_\to \mathfrak{U}_\gamma)$ is strongly continuous.

Proof of Assertion 4: Therefore let $(g_n)_{n\in I} \subset G$ be a net with $\lim g_n = g$. We now want to show:

$$\forall\epsilon > 0 : \exists N \in I : \forall n \geq N : \|\Phi(g_n, a) - \Phi(g, a)\| < \epsilon$$

Therefore let $(a_m)_{m\in\mathbb{N}} \subset \bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma)$ be a sequence with $\lim_{m\to\infty} a_m = a \in \lim \mathfrak{U}_\gamma$. Now let $m \in \mathbb{N}$ arbitrary. Then:

$$\begin{aligned}
&\|\Phi(g_n, a) - \Phi(g, a)\| \\
&= \|\Phi(g_n, a) - \Phi(g_n, a_m) + \Phi(g_n, a_m) - \Phi(g, a_m) + \Phi(g, a_m) - \Phi(g, a)\| \\
&\leq \|\Phi(g_n, a) - \Phi(g_n, a_m)\| + \|\Phi(g_n, a_m) - \Phi(g, a_m)\| + \|\Phi(g, a_m) - \Phi(g, a)\| \qquad (4.4)
\end{aligned}$$

Now consider the first summand of eq. (4.4). Let $\epsilon > 0$. Since $*$-morphisms are contracting and since (a_m) converges towards a we have:

$$\|\Phi(g_n, a) - \Phi(g_n, a_m)\| = \|\Phi(g_n, a - a_m)\| \leq \|a - a_m\| < \frac{\epsilon}{3}$$

Now consider the third summand. For the same ϵ with the same M we have with the same argumentation also for all $m \geq M$:

$$\|\Phi(g, a_m) - \Phi(g, a)\| = \|\Phi(g, a_m - a)\| \leq \|a - a_m\| < \frac{\epsilon}{3}$$

We now consider the second summand. Therefore choose $m \geq M$ arbitrary but fixed. Then there is a $\gamma \in L$ with $a_m \in \mathrm{im}(\phi_\gamma)$. Set $a_m^{(\gamma)}$ such that $\phi_\gamma\left(a_m^{(\gamma)}\right) = a_m$. Then there is an $N \in I$ such that for all $n \geq N$ the following holds:

$$\begin{aligned}
\Phi(g_n, a_m) - \Phi(g, a_m) &= \|\Phi(g_n, \phi_\gamma(a_m^{(\gamma)})) - \Phi(g, \phi_\gamma(a_m^{(\gamma)}))\| \\
&= \|\phi_\gamma(g_n a_m^{(\gamma)}) - \phi_\gamma(g a_m^{(\gamma)})\| \\
&= \|\phi_\gamma(g_n a_m^{(\gamma)} - g a_m^{(\gamma)})\| \\
&= \|g_n a_m^{(\gamma)} - g a_m^{(\gamma)}\| < \frac{\epsilon}{3}
\end{aligned}$$

Where we have used continuity of Φ_γ in pointwise norm topology. And hence we have shown, that:

$$\forall\epsilon > 0 \exists N \in I : \forall n \geq N : \|\Phi(g_n, a) - \Phi(g, a)\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

Assertion 5: Φ is really the inductive limit C^*-group action.

Proof of Assertion 5: It follows directly by the definition of Φ, that $\phi_\gamma \circ \Phi_\gamma(\cdot, g) = \Phi(\cdot, g) \circ \phi_\gamma$. Hence it is the inductive limit group action.

Assertion 6: The inductive limit C^*-group action is unique.

Proof of Assertion 6: Assume that there is another inductive limit C^*-group action $\Psi : G \curvearrowright \lim_{\rightarrow} \mathfrak{U}_\gamma$ with $\Psi \neq \Phi$. This means especially, that there is a $g \in G$ with $\Psi_g \neq \Phi_g$. By the uniqueness of the bounded extension given by lemma 9 this would mean, that

$$\Phi_g|_{\bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma)} \neq \Psi_g|_{\bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma)} \tag{4.5}$$

We now have, since Φ and Ψ are inductive limit C^*-group actions, that for $a_\gamma \in \mathrm{im}(\phi_\gamma)$ the following holds:

$$\Phi_g(a_\gamma) = \phi_\gamma(ga_\gamma) = \Psi_g(a_\gamma)$$

Which is a contradiction to eq. (4.5). $\square$

We now want to investigate the prototypical example of the algebra of continuous functions in the context of inductive limits:

Theorem 32
Let $(X_\gamma, p_{\gamma\gamma'})_{\gamma,\gamma'\in L}$ be a projective family of compact Hausdorff spaces with projective limit $(\lim_{\leftarrow} X_\gamma, (p_\gamma)_{\gamma\in L})$ together with a compatible family of group actions $\tilde{\Phi}_\gamma : X_\gamma \curvearrowleft G$ with projective limit $\tilde{\Phi} : \lim_{\leftarrow} X_\gamma \curvearrowleft G$. Then:

1. $\Big((C(X_\gamma), G, \Phi_\gamma), \phi_{\gamma'\gamma} = p^*_{\gamma\gamma'}\Big)_{\gamma,\gamma'\in L}$ *with*

$$\Phi_\gamma : G \times C(X_\gamma) \to C(X_\gamma), (g, f_\gamma) \mapsto f_\gamma \circ \tilde{\Phi}_\gamma(\cdot, g)$$

is an inductive family of C^*-dynamical systems.

2. *Its inductive limit is given by the C^*-dynamical system $(C(\lim_{\leftarrow} X_\gamma), G, \Phi)$ together with the maps $(\phi_\gamma = p^*_\gamma)_{\gamma\in L}$ where*

$$\Phi : G \times C(\lim_{\leftarrow} X_\gamma) \to C(\lim_{\leftarrow} X_\gamma), (g, f) \mapsto f \circ \tilde{\Phi}(\cdot, g)$$

Proof. 1.) We've already shown in lemma 21, that $\forall\gamma \in L$ it holds, that $(C(X_\gamma), G, \Phi_\gamma)$ is a C^*-dynamical system. Further we have shown in theorem 20, that $(C(X_\gamma), \phi_{\gamma'\gamma})$ is an inductive family of C^*-algebras. Hence we have still to show, that the C^*-group actions constitute an inductive family of C^*-group actions, i.e., that $\phi_{\gamma'\gamma}(g\cdot) = g\phi_{\gamma'\gamma}(\cdot)$ holds for all $\gamma' \geq \gamma \in L$. Therefore let $\gamma \in L$ and $f \in C(X_\gamma)$. Then:

$$\begin{aligned}
\phi_{\gamma'\gamma}(gf) &= \phi_{\gamma'\gamma}(f \circ \tilde{\Phi}_\gamma(\cdot, g)) \\
&= f \circ \tilde{\Phi}_\gamma(\cdot, g) \circ p_{\gamma\gamma'} \\
&= f \circ p_{\gamma\gamma'} \circ \tilde{\Phi}_{\gamma'}(\cdot, g) \\
&= g\phi_{\gamma'\gamma} f
\end{aligned}$$

And hence the claim follows.
2.) Please recall that the inductive limit group action, which we will denote by Φ, is defined on the dense subset $\bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma)$ as

$$\forall g \in G : \Phi_g : \bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma) \to \bigcup_{\gamma\in L} \mathrm{im}(\phi_\gamma), \phi_\gamma(a_\gamma) \mapsto \phi_\gamma(ga_\gamma)$$

Hence in our case:

$$\forall g \in G : \Phi : \bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma) \to \bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma), f_\gamma \circ p_\gamma \mapsto f_\gamma \circ \tilde{\Phi}_\gamma(\cdot, g) \circ p_\gamma$$

Now we have by G-equivariance of p_γ:

$$f_\gamma \circ \tilde{\Phi}_\gamma(\cdot, g) \circ p_\gamma = f_\gamma \circ p_\gamma \circ \tilde{\Phi}_g$$

I.e.

$$\forall g \in G : \Phi_g : \bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma) \to \bigcup_{\gamma \in L} \mathrm{im}(\phi_\gamma), f_\gamma \circ p_\gamma \mapsto f_\gamma \circ p_\gamma \circ \tilde{\Phi}_g = \Phi_g(f_\gamma \circ p_\gamma)$$

Hence we have shown, that Φ satisfies the defining property of an inductive limit C^*-group action on a dense subset. Hence the claim follows by uniqueness of the extension. □

4.5. Spectral Theory for Inductive Limits of C^*-Dynamical Systems

Corollar 6
Let $(X_\gamma, p_{\gamma\gamma'})_{\gamma.\gamma' \in L}$ be a projective family of compact Hausdorff spaces with projective limit $(\lim_\leftarrow X_\gamma, (p_\gamma)_{\gamma \in L})$. Let $\tilde{\Phi}_\gamma : X_\gamma \curvearrowleft G$ be a compatible family of group actions with projective limit group action $\tilde{\Phi} : \lim_\leftarrow X_\gamma \curvearrowleft G$. Let further $\big((C(X_\gamma), G, \Phi_\gamma)\phi_{\gamma'\gamma}\big)_{\gamma,\gamma' \in L}$ be the corresponding family of C^*-dynamical systems as defined in the last theorem.
Then:

1. $\Big(\mathrm{Fix}_G(C(X_\gamma)), \phi_{\gamma'\gamma}\big|_{\mathrm{Fix}_G(C(X_\gamma))}\Big)_{\gamma,\gamma' \in L}$ *is an inductive family of C^*-algebras.*
2. $\lim_\to \mathrm{Fix}_G\left(C(X_\gamma)\right) = \mathrm{Fix}_G\left(\lim_\to C(X_\gamma)\right)$.

Proof. 1.) This assertion will be shown in theorem 33 for general unital, abelian C^*-algebras without any reference to this claim. Hence this claim follows as a special case.
2.) We have already shown in lemma 21 and theorem 32, that $(C(X_\gamma), G, \Phi_\gamma)$ and $(\lim_\to C(X_\gamma), G, \Phi)$ are abelian, unital C^*-dynamical systems. Hence it follows with lemma 19, that $\mathrm{Fix}_G(\lim_\to C(X_\gamma))$ is an abelian, unital C^*-algebra. We know further from lemma 21 that

$$\mathrm{Fix}_G(C(X_\gamma)) = C(X_\gamma/G)$$

Further we have with theorem 30 and theorem 20, that:

$$\lim_{\leftarrow}(X_\gamma/G) = (\lim_{\leftarrow} X_\gamma)/G$$
$$\lim_{\to} C(X_\gamma/G) = C(\lim_{\leftarrow} X_\gamma/G)$$
$$\lim_{\to} C(X_\gamma) = C(\lim_{\leftarrow} X_\gamma)$$

All together:

$$\mathrm{Fix}_G(\lim_{\to} C(X_\gamma)) = \mathrm{Fix}_G(C(\lim_{\leftarrow} X_\gamma)) = C(\lim_{\leftarrow} X_\gamma/G)$$

And further:

$$\lim_{\to} \mathrm{Fix}_G(C(X_\gamma)) = \lim_{\to} C(X_\gamma/G) = C((\lim_{\leftarrow} X_\gamma)/G) = \mathrm{Fix}_G(\lim_{\to} C(X_\gamma))$$

And the claim is shown. □

We now generalize this result:

Theorem 33
Let $\left((\mathfrak{U}_\gamma, G, \Phi_\gamma), \phi_{\gamma'\gamma}\right)_{\gamma,\gamma' \in L}$ be an inductive family of C^*-dynamical systems such that $\mathfrak{U}_\gamma$ is abelian and unital for each $\gamma \in L$. Then:

1. $\left(\mathrm{Fix}_G(\mathfrak{U}_\gamma), \phi_{\gamma'\gamma}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)}\right)_{\gamma' \geq \gamma \in L}$ *is an inductive family of abelian, unital C^*-algebras.*
2. $\lim_\to \mathrm{Fix}_G(\mathfrak{U}_\gamma) = \mathrm{Fix}_G(\lim_\to \mathfrak{U}_\gamma)$

Proof. 1.) We have already shown, that $\mathrm{Fix}_G(\mathfrak{U}_\gamma)$ is an abelian, unital C^*-algebra for all $\gamma \in L$. We now have to show, that $\phi_{\gamma'\gamma}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)}$ is a well defined isometric $*$-morphism and satisfies the composition properties demanded in the definition of an inductive family:

$$\phi_{\gamma\gamma}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)} = id.$$
$$\phi_{\gamma''\gamma'}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)}\, \phi_{\gamma'\gamma}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)} = \phi_{\gamma''\gamma}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)}$$

But therefore observe, that both is trivially satisfied, since $\phi_{\gamma'\gamma}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)}$ an isometric $*$-morphism that satisfies the composition properties of an inductive family restricted to a subalgebra. It remains to show well definedness.

Therefore let $a_\gamma \in \mathrm{Fix}_G(\mathfrak{U}_\gamma)$ and $g \in G$. Then:

$$\begin{aligned} g \cdot \phi_{\gamma'\gamma}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)}(a_\gamma) &= g \cdot \phi_{\gamma'\gamma}(a_\gamma) \\ &= \phi_{\gamma'\gamma}(g a_\gamma) \\ &= \phi_{\gamma'\gamma}(a_\gamma) \\ &= \phi_{\gamma'\gamma}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)}(a_\gamma) \end{aligned}$$

and hence $\mathrm{im}\left(\phi_{\gamma'\gamma}|_{\mathrm{Fix}_G(\mathfrak{U}_\gamma)}\right) \subseteq \mathrm{Fix}_G(\mathfrak{U}_{\gamma'})$.
2.) Let $\mathcal{G}_\gamma : \mathfrak{U}_\gamma \to C(\Delta(\mathfrak{U}_\gamma))$ and $\mathcal{G} : \lim_\to \mathfrak{U}_\gamma \to C(\Delta(\lim_\to \mathfrak{U}_\gamma))$ be the corresponding Gel'fand transforms, which are isometric isomorphisms of C^*-algebras. Since $\mathfrak{U}_\gamma \cong C(\Delta(\mathfrak{U}_\gamma))$ we have $\lim_\to \mathrm{Fix}_G(\mathfrak{U}_\gamma) \cong \lim_\to \mathrm{Fix}_G(C(\Delta(\mathfrak{U}_\gamma)))$. Further we have by corollar 6, that $\lim_\to \mathrm{Fix}_G(C(\Delta(\mathfrak{U}_\gamma))) = \mathrm{Fix}_G(\lim_\to C(\Delta(\mathfrak{U}_\gamma)))$. Now since $\lim_\to C(\Delta(\mathfrak{U}_\gamma)) \cong \lim_\to \mathfrak{U}_\gamma$ via the Gel'fand transform, we have finally $\mathrm{Fix}_G(\lim_\to C(\Delta(\mathfrak{U}_\gamma))) \cong \mathrm{Fix}_G(\lim_\to \mathfrak{U}_\gamma)$, and the assertion follows. □

All together we have now the following theorem:

Theorem 34
Let $\left((\mathfrak{U}_\gamma, G, \Phi_\gamma), \phi_{\gamma'\gamma}\right)_{\gamma,\gamma' \in L}$ be an inductive family of C^*-dynamical systems. Then:

$$\Delta\left(\mathrm{Fix}_G(\lim_{\to} \mathfrak{U}_\gamma)\right) = \lim_{\leftarrow} \left(\Delta(\mathfrak{U}_\gamma)/G\right)$$

Proof. This is an easy corollar of preceding theorems. By theorem 33 we have:

$$\Delta\left(\mathrm{Fix}_G(\lim_{\to} \mathfrak{U}_\gamma)\right) = \Delta\left(\lim_{\to} \mathrm{Fix}_G(\mathfrak{U}_\gamma)\right)$$

By theorem 18 we have:

$$\Delta\left(\lim_{\to} \mathrm{Fix}_G(\mathfrak{U}_\gamma)\right) = \lim_{\leftarrow} \Delta(\mathrm{Fix}_G(\mathfrak{U}_\gamma))$$

By theorem 28 we have further:

$$\Delta(\mathrm{Fix}_G(\mathfrak{U}_\gamma)) = \Delta(\mathfrak{U}_\gamma)/G$$

And this gives all together:

$$\Delta\left(\mathrm{Fix}_G(\lim_{\rightarrow} \mathfrak{U}_\gamma)\right) = \lim_{\leftarrow} \left(\Delta(\mathfrak{U}_\gamma)/G\right)$$

And hence the claim follows. $\square$

Part II.

Application to Polymer and Loop Quantization

5. Application to Polymer Quantization of the Scalar Field

The aim of this chapter is to calculate the quantum configuration space of polymer matter using the methods developed in this thesis. Therefore in the first section a new set of configuration variables for scalar field theory is given, which will be used for the quantization procedure. The second section defines the corresponding quantum algebra. The third section expresses this quantum algebra as an inductive limit algebra using two nested inductive limit procedures. In the fourth section the spectral theory of the elementary constituents of the inductive family is investigated. In the next section then finally the quantum configuration space of the full point holonomy algebra is computed using the methods presented in chapter 3. The last section comprises a discussion of our approach.

The definition of the classical configuration variables as well as the definition of the quantum algebra orient themselves at [20]. However our definitions differ slightly from the definitions presented in [20], since our constructions are somewhat more convenient for the inductive limit procedure. Anyhow the reader is encouraged to note, that the constructions presented in [20] are equivalent to the constructions used in this thesis. The topological considerations appearing in the fifth section can be found partly in Ch. 28 of [31]. The corresponding inductive limit construction on the C^*-side as well as the correspondence between the latter and the projective limit construction on the topological side is firstly investigated in this thesis. The application of this correspondence to the calculation of the full quantum configuration space is also presented firstly in this thesis.

Finally the author wants to emphasize that polymer quantization serves as an easy toy example for understanding loop quantization, since the algebraic structure of theory is very similiar to the corresponding structures in loop quantum gravity. However, the system of a scalar field comprises no gauge redundancies, and hence the theory is somewhat simpler. The latter fact also expresses himself in the fact, that in this Chapter no results of chapter 4 are needed.

5.1. Classical Configuration Variables: Point Holonomies

Let M be a compact, smooth, pseudoriemannian Manifold and let $\vartheta : \mathbb{R} \times \Sigma \to M$ be a foliation by euclidean hypersurfaces. The classical configuration space of a real scalar thereon is then given by:

$$\mathcal{A}_{class} = \{\phi \in C_c^\infty(\Sigma, \mathbb{R})\}$$

We now define the a new set of configuration variables, which will serve as a starting point for the so called polymer quantization:

Definition 28 (cp. [20])
Let $\lambda : \Sigma \to \mathbb{R}$ be a function with finite support. Then define the point holonomy corre-

sponding to λ as:

$$N_\lambda : \mathcal{A} \to \mathbb{C}, \phi \mapsto e^{i\sum_{v\in\mathrm{supp}(\lambda)} \lambda(v)\phi(v)}$$

Please observe that the Poisson bracket of two point holonomies vanishes and additionally further relations hold:

Lemma 25
Let anything be as before. Then the Poisson bracket of two point holonomies vanishes, i.e.:

$$\{N_\alpha, N_\beta\} = 0$$

Further the following relations hold:

$$\begin{aligned} N_\alpha N_\beta &= N_{\alpha+\beta} \\ \overline{N}_\alpha &= -N_\alpha \\ N_\alpha = \mathbb{1}_{\mathcal{A}} &\Leftrightarrow \alpha = 0 \end{aligned}$$

where $\mathbb{1}_{\mathcal{A}} : \mathcal{A} \to \mathbb{R}, \phi \mapsto 1$ and where $\overline{N}_\alpha$ denotes the complex conjugate of N_α.

Proof. That the Poisson bracket vanishes follows directly, since the point holonomy does not depend on the momentum variables. The other relations can also be shown straightforwardly. Observe further, that $N_0 = \mathbb{1}_{\mathcal{A}}$ follows directly, since $e^0 = 1$. Now observe, that the other direction follows, since $N_\alpha = 0$ implies

$$\forall \phi \in \mathcal{A} : \exists k \in \mathbb{Z} : \sum_{v\in\mathrm{supp}(\alpha)} \alpha(v)\phi(v) = 2\pi k$$

Then $\forall v \in \mathrm{supp}(\alpha) : \alpha(v) = 0$ can be shown by choosing an arbitrary $v \in \mathrm{supp}(\alpha)$ and a $\phi \in \mathcal{A}$ which vanishes everywhere except for a small neighborhood $v \in U$ with $\mathrm{supp}(\alpha) \cap U = \{v\}$. □

5.2. The Point Holonomy Algebra

We now want to define the quantum algebra of configuration variables which will be called the point holonomy algebra. We orient ourselves at the definition used in [20], but our definition is slightly different, since we used a different, but equivalent, definition for the classical point holonomies.

Definition 29 (cp. [20])
Let $\mathcal{C} = \{N_\lambda | \lambda : \Sigma \to \mathbb{R}$ has finite support$\}$ and let $F\mathcal{C}$ be the free complex vector space over $\mathcal{C}$. Please note that at this sate N_λ is just a symbol used for the definition of the free vector space.

Define a multiplication on $F\mathcal{C}$ via

$$\left(\sum_{i=1}^{n} z_i N_{\lambda_i}\right) \cdot \left(\sum_{j=1}^{m} z_j N_{\lambda_j}\right) := \sum_{i=1}^{n}\sum_{j=1}^{m} z_i z_j N_{\lambda_i+\lambda_j}$$

and a $*$-structure is defined by

$$\left(\sum_{i=1}^{n} z_i N_{\lambda_i})\right)^* = \sum_{i=1}^{n} \bar{z}_i N_{-\lambda_i}$$

and define further a norm by:

$$\left\|\sum_{i=1}^{n} z_i N_{\lambda_i}\right\| = \sup_{\phi\in C_c(\Sigma,\mathbb{R})} \left|\sum_{i=1}^{n} z_i N_{\lambda_i}[\phi]\right|$$

Then define the point holonomy algebra as the abelian, unital C^*-algebra $\overline{\mathfrak{C}}$ with $\mathfrak{C} = F\mathcal{C}$ where the completion is performed with respect to above norm and further $\overline{\mathfrak{C}}$ is endowed with the $*$-algebra structure from above.

We now show, that this really defines an abelian, unital C^* algebra as proposed in the definition:

Lemma 26
The point holonomy algebra $\overline{\mathfrak{C}}$ defines an abelian, unital C^*-algebra with unit $\mathbb{1} = N_0$ where $0 : \Sigma \to \mathbb{R}, x \mapsto 0$ (have in mind, that $\operatorname{supp}(0) = \emptyset$ and hence is finite).

Proof. We first show, that the multiplication is associative. With lemma 37 we have, that it is sufficient to show, that $(N_\alpha N_\beta)\, N_\gamma = N_\alpha\,(N_\beta N_\gamma)$ holds. This is indeed the case, since:

$$\begin{aligned}(N_\alpha N_\beta)\, N_\gamma &= N_{(\alpha+\beta)+\gamma}\\ N_\alpha\,(N_\beta N_\gamma) &= N_{\alpha+(\beta+\gamma)}\end{aligned}$$

Here we have used, that $+$ is associative on $\mathbb{R}$ and hence associative on real valued functions. Now we show, that $*$ is an involution. With lemma 37 we have, that it is sufficient to show, that $N_\alpha^{**} = N_\alpha$ and $(N_\alpha N_\beta)^* = N_\beta^* N_\alpha^*$ hold. This is indeed the case, since:

$$\begin{aligned}N_\alpha^{**} &= N_{-\alpha}^* = N_\alpha\\ (N_\alpha N_\beta)^* &= N_{\alpha+\beta}^* = N_{-\alpha-\beta} = N_{-\beta-\alpha} = N_\beta^* N_\alpha^*\end{aligned}$$

We now show, that the norm is submultiplicative and satisfies the C^*-property. This follows by the usual analytical tricks:

$$\begin{aligned}\left\|\sum_{i=1}^{n} z_i N_{\alpha_i} \sum_{j=1}^{m} \tilde{z}_j N_{\beta_j}\right\| &= \sup_{\phi\in C_c^\infty(\Sigma)} \left|\sum_{i=1}^{n} z_i N_{\alpha_i}[\phi] \sum_{j=1}^{m} \tilde{z}_j N_{\beta_j}[\phi]\right|\\ &\leq \sup_{\phi_2\in C_c^\infty(\Sigma)} \left(\sup_{\phi_1\in C_c^\infty(\Sigma)} \left|\sum_{i=1}^{n} z_i N_{\alpha_i}[\phi_1]\right| \left|\sum_{j=1}^{m} \tilde{z}_j N_{\beta_j}[\phi_2]\right|\right)\\ &= \left\|\sum_{i=1}^{n} z_i N_{\alpha_i}\right\| \left\|\sum_{j=1}^{m} \tilde{z}_j N_{\beta_j}\right\|\end{aligned}$$

and

$$\begin{aligned}\left\| \left(\sum_{i=1}^{n} z_i N_{\alpha_i}\right)\left(\sum_{i=1}^{n} z_i N_{\alpha_i}\right)^* \right\| &= \sup_{\phi\in C_c^\infty(\Sigma)} \left| \left(\sum_{i=1}^{n} z_i N_{\alpha_i}[\phi]\right) \overline{\left(\sum_{i=1}^{n} z_i N_{\alpha_i}[\phi]\right)} \right| \\ &= \left(\sup_{\phi\in C_c^\infty(\Sigma)} \left|\sum_{i=1}^{n} z_i N_{\alpha_i}[\phi]\right|\right)^2 \\ &= \left\|\sum_{i=1}^{n} z_i N_{\alpha_i}\right\|^2\end{aligned}$$

Finally we have, that the algebra is commutative, since

$$N_\alpha N_\beta = N_{\alpha+\beta} = N_{\beta+\alpha} = N_\beta N_\alpha$$

and has unit $\mathbb{1} = N_0$ since:

$$N_0 N_\alpha = N_\alpha = N_\alpha N_0$$

Hence $\mathfrak{C}$ is an abelian, unital pre C^*-algebra by lemma 37 and hence $\overline{\mathfrak{C}}$ is an abelian, unital C^*-algebra. □

5.3. The Point Holonomy Algebra as an Inductive Limit C^*-Algebra

In this section we will see, that the point holonomy algebra can be expressed by using two nested inductive limit constructions:

5.3.1. The Point Holonomy Algebra as an Inductive Limit of C^*-Algebras of Almost Periodic Functions

Define the following objects:

Definition 30
Let anything be as before. Then:

1. $L_1 := \{V \subset \Sigma | V \text{ is finite}\}$ *together with the relation* $V \leq V' \Leftrightarrow V \subseteq V'$.
2. *For each* $V \in L_1$ *define the algebra* $\overline{\mathfrak{C}}_V$ *via* $\mathfrak{C}_V = F\mathcal{C}_V$ *with*

$$\mathcal{C}_V = \{N_\lambda | \lambda : \Sigma \to \mathbb{R} \wedge \operatorname{supp}(\lambda) \subseteq V\}$$

   and C^**-algebra structure as defined for* $\overline{\mathfrak{C}}$.
3. *For each* $V \leq V'$ *define* $\phi_{V'V} : \mathfrak{C}_V \to \mathfrak{C}_{V'}$ *via* $N_\lambda \mapsto N_\lambda$.

We now can show the following:

Theorem 35
Let anything be as before. Then:

1. L_1 *is a label set.*
2. *For each* $V \in L$ *the map* $i : \overline{\mathfrak{C}}_V \hookrightarrow \overline{\mathfrak{C}}, N_\alpha \mapsto N_\alpha$ *defines an isometric embedding.*

3. *$(\overline{\mathfrak{C}}_V, \phi_{V'V})_{V,V'\in L}$ is an inductive family of abelian, unital C^*-algebras with unit N_0.*

4. *$\overline{\mathfrak{C}}$ is the inductive limit of this inductive family.*

Proof. 1.) We have reflexivity since $V_1 \subset V_1$. Further we have symmetry, since $(V_1 \subset V_2) \wedge (V_2 \subset V_1) \Rightarrow V_1 = V_2$. Transitivity also follows easily, since $(V_1 \subset V_2) \wedge (V_2 \subset V_3) \rightarrow (V_1 \subset V_3)$. That the set is directed follows also, since we can set $\hat{V} = V_1 \cup V_2$ for $V_1, V_2 \in L$. With this $V_1, V_2 \subset \hat{V}$ follows.
2.) Define the conjectured isometric embedding

$$i : \overline{\mathfrak{C}}_V \hookrightarrow \overline{\mathfrak{C}}$$

via its action on basis elements as $N_\alpha \mapsto N_\alpha$. Observe that

$$\begin{aligned} i(N_\alpha N_\beta) = i(N_{\alpha+\beta}) = N_{\alpha+\beta} = N_\alpha N_\beta = i(N_\alpha) i(N_\beta) \\ i(N_\alpha^*) = i(N_{-\alpha}) = N_{-\alpha} = N_\alpha^* = i(N_\alpha)^* \end{aligned}$$

and hence it defines a unique $*$-morphism by lemma 38. Further observe that it is an isometry:

$$\begin{aligned} \left\| i\left(\sum_{i=1}^n z_i N_{\alpha_i}\right)\right\| &= \sup_{\phi\in C_c^\infty(\Sigma)} \left|\sum_{i=1}^n z_i N_{\alpha_i}[\phi]\right| \\ &= \left\|\sum_{i=1}^n z_i N_{\alpha_i}\right\| \end{aligned}$$

Hence the claim follows.
3.) That $\overline{\mathfrak{C}}_V$ is an commutative C^*-algebra follows directly with 2.). Further it is unital for any V, since $\operatorname{supp}(0) = \emptyset \subset V$. We now want to show, that $\phi_{V'V}$ is an isometric $*$-morphism and satisfies the composition properties needed for $(\overline{\mathfrak{C}}_V, \phi_{V'V})_{V,V'\in L}$ to be an inductive family. That it defines an isometric $*$-morphism is shown in the same way, as it was shown, that i defines an isometric $*$-morphism. To show the composition properties demanded in the definition of an inductive family observe the following:

$$\begin{aligned} &\phi_{V_3V_2}\phi_{V_2V_1} : N_\alpha \mapsto N_\alpha \\ &\phi_{V_3V_1} : N_\alpha \mapsto N_\alpha \\ &\phi_{VV} : N_\alpha \mapsto N_\alpha \\ &id. : N_\alpha \mapsto N_\alpha \end{aligned}$$

Since by lemma 38 and lemma 9 an isometric $*$-morphism is uniquely defined by its action on the basis elements we hence obtain:

$$\begin{aligned} \phi_{V_3V_2}\phi_{V_2V_1} &= \phi_{V_3V_1} \\ \phi_{VV} &= id. \end{aligned}$$

Hence the claim follows.
4.) We use lemma 11. First observe, that $\overline{\mathfrak{C}}_V \subset \overline{\mathfrak{C}}$ is a subalgebra as it was shown in 2.). Further observe, that $\overline{\mathfrak{C}}_V \subset \overline{\mathfrak{C}}_{V'}$ for $V \leq V'$ can be considered as a subalgebra since $\phi_{V'V}$ is an injective $*$-morphism. Hence we have:

$$\lim_{\rightarrow} \overline{\mathfrak{C}}_V = \overline{\bigcup_{V\in L} \overline{\mathfrak{C}}_V} = \overline{\bigcup_{V\in L} \mathfrak{C}_V}$$

Then the claim follows with lemma 11 if the following holds:

$$\bigcup_{V\in L}\mathfrak{C}_V=\mathfrak{C}$$

But this is indeed the case. Therefore let $N=\sum_{i=1}^n \lambda_i N_{\alpha_i}\in\bigcup_{V\in L}\mathfrak{C}_V$. Then each α_i has finite support and hence N lies in $\mathfrak{C}$. On the other hand let $\sum_{i=1}^n \lambda_i N_{\alpha_i}\in\mathfrak{C}$. Then choose $V=\bigcup_{i=1}^n \operatorname{supp}(\alpha_i)$, which is finite, since it is a finite union of finite sets. Hence $V\in L$ and $N\in\mathfrak{C}_V\subset\bigcup_{V\in L}\mathfrak{C}_V$. □

The elementary constituents of this inductive family are algebras of the form $\overline{\mathfrak{C}}_{\{x\}}$ for $x\in\Sigma$. Since the basis elements of this algebra are given by $(N_\lambda)_{\lambda\in\mathbb{R}}$, a finite linear combination of those corresponds to a classical observable of the type:

$$\phi\mapsto\sum_{i=1}^n z_i e^{i\lambda_i\phi(x)}$$

Such a function can be seen as a superposition of trigonometric functions and the closure of all such functions is called the algebra of Bohr almost periodic functions.

5.3.2. The C^*-Algebra of Almost Periodic Functions as an Inductive Limit C^*-algebra

In this section we want to express the algebra $\overline{\mathfrak{C}}_{\{x\}}$ as an inductive limit of more elementary C^*-algebras whose spectra can be calculated by easy algebraic methods.

Therefore we first define the corresponding label set:

Definition 31 (cp. Ch. 28 of [31], [34]) *1. A tuple $(k_i)_{i=1}^n\subset\mathbb{R}$ is called rationally independent, if for $q_i\in\mathbb{Q}$ it holds, that:*

$$\sum_{i=1}^n q_i k_i=0\Rightarrow\forall i:q_i=0$$

2. Define the additive subgroup of $\mathbb{R}$ generated by a finite set of rationally independent frequencies $(k_i)_{i=1}^n$ as

$$S(k_1,...,k_n)=\left\{\sum_{i=1}^n q_i k_i|q_i\in\mathbb{Q}\right\}$$

3. Define the label set $(L_2,\leq)$ via

$$L_2=\{S(k_1,...,k_n)|k_i\in\mathbb{R},n\in\mathbb{N}\}$$

and $S_1\leq S_2\in L_2$ iff for $S_1=S(k_1,...,k_n)$ and $S_2=S(h_1,...,h_m)$ it holds, that there are $q_{ij}\in\mathbb{Z}$ with $k_i=\sum_{j=1}^m q_{ij}h_j$.

Before showing, that this really constitutes a label set, we want to define the corresponding inductive family of C^*-algebras:

Definition 32
Let $x\in\Sigma$.

1. *Let $(k_i)_{i=1}^{n_S}$ be a tuple of n_S rationally independent numbers and let $S = S(k_1, ..., k_{n_S})$ be the corresponding additive subgroup of $\mathbb{R}$. Then define $\overline{\mathfrak{C}}_x(S)$ via $\mathfrak{C}_x(S) = F\mathcal{C}_x(S)$ and:*
$$\mathcal{C}_x(S) = \{N_\lambda | \lambda : \Sigma \to S \wedge \operatorname{supp}(\lambda) \subset \{x\}\} \cong S$$
Define further a norm and a $*$-structure for $\overline{\mathfrak{C}}_x(S)$ analogously as in definition 29.

2. *For convenience denote the abelian, unital C^*-algebra $\overline{\mathfrak{C}}_{\{x\}}$ by $\overline{\mathfrak{C}_x} = \overline{\mathfrak{C}_{\{x\}}}$.*

We now show, that all this is well defined:

Lemma 27
Let anything be as before. Then:

1. *L_2 is a Label set.*
2. *$i : \overline{\mathfrak{C}}_x(S) \hookrightarrow \overline{\mathfrak{C}}_x$ is an isometric embedding of C^*-algebras.*
3. *$\overline{\mathfrak{C}}_x(S)$ is an abelian, unital C^*-algebra.*
4. *The tuple $(\overline{\mathfrak{C}_x(S)}, \Lambda_{S'S})_{S',S \in L_2}$ is an inductive system of C^*-algebras.*
5. *$\lim_\to \overline{\mathfrak{C}_x(S)} = \overline{\mathfrak{C}}_x$.*

Proof. 1.) First reflexivity is shown: $S(k_1, ..., k_n) \leq S(k_1, ..., k_n)$ holds, since:
$$k_i = \sum_{i=1}^{n} \delta_{ij} k_j$$
Now towards symmetry: Let $S(k_1, ..., k_n) \leq S(h_1, ..., h_m)$ and $S(h_1, ..., h_m) \leq S(k_1, ..., k_m)$. This means that there are $q_{ij}, \tilde{q}_{ij} \in \mathbb{Z}$ with:
$$k_i = \sum_{j=1}^{m} q_{ij} h_j$$
$$h_i = \sum_{j=1}^{n} \tilde{q}_{ij} h_j$$
Now let $\sum_{i=1}^n q_i k_i \in S(k_1, ..., k_n)$. Then:
$$\sum_{i=1}^{n} q_i k_i = \sum_{i=1}^{n} \sum_{j=1}^{m} q_i q_{ij} h_j = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} q_i q_{ij} \right) h_j$$
Since $(\sum_{i=1}^n q_i q_{ij}) \in \mathbb{Q}$, it follows then, that $\sum_{i=1}^n q_i k_i \in S(h_1, ..., h_m)$. The inclusion is shown in the same way, and hence $S(k_1, ..., k_n) = S(h_1, ..., h_m)$. Now towards transitivity: Let $S(k_1, ..., k_n) \leq S(h_1, ..., h_m)$ and $S(h_1, ..., h_m) \leq S(r_1, ..., r_s)$. I.e.:
$$k_i = \sum_{j=1}^{m} q_{ij} h_j$$
$$h_j = \sum_{l=1}^{s} \tilde{q}_{jl} r_l$$

Hence $k_i = \sum_{j=1}^m \sum_{l=1}^s q_{ij}\tilde{q}_{jl} = \sum_{l=1}^s \left(\sum_{j=1}^m q_{ij}\tilde{q}_{jl}\right) r_l$ and hence $S(k_1, ..., k_n) \leq S(r_1, ..., r_s)$.

Finally we want to show, that $\mathcal{L}$ together with $\leq$ is directed. Consider $S = S(k_1, ..., k_n)$ and $S' = S(h_1, ..., h_m)$. Now set $\hat{S} = S(r_1, ..., r_{n+m})$ with:

$$r_i = \begin{cases} k_i & i = 1, ..., n \\ h_{i-n} & i = n+1, ..., n+m \end{cases}$$

Then

$$k_i = \sum_{j=1}^{m+n} q_{ij} r_j$$

$$h_i = \sum_{j=1}^{m+n} \tilde{q}_{ij} r_j$$

holds, with

$$q_{ij} = \begin{cases} \delta_{ij} & j = 1, ..., n \\ 0 & j = n+1, ..., n+m \end{cases}$$

and

$$\tilde{q}_{ij} = \begin{cases} 0 & j = 1, ..., n \\ \delta_{i(j-n)} & j = n+1, ..., n+m \end{cases}$$

and hence $S', S \leq \hat{S}$ and the assertion follows.
2.) This is shown in totally the same way, as assertion 2 of theorem 35 was shown. 3.) That it is an abelian C^*-algebra follows directly with 2.). That it is unital follows, since $0 \in S$ and hence $N_0 \in \overline{\mathfrak{C}_x(S)}$.
4.) That each $\overline{\mathfrak{C}_x(S)}$ is a C^*-algebra was already shown above. That for $S' \geq S$ it holds, that $\Lambda_{S'S}$ is an isometric $*$-morphism and satisfies the composition properties is shown in the same way as in theorem 35.
5.) For this we use again lemma 11. Therefore first observe, that $\mathfrak{C}_x(S) \subset \mathfrak{C}_x$ follows with 2.). Now let $S' \geq S \in L$. I.e. there is a $q_{ij} \in \mathbb{Z}$ s.th.

$$k_i = \sum_j q_{ij} h_j$$

holds for $S = S(k_1, ..., k_n)$ and $S' = S(h_1, ..., h_m)$. Hence we have for $\lambda \in S$, that

$$\lambda = \sum_i q_i k_i = \sum_{ij} q_i q_{ij} h_j = \sum_j \left(\sum_i q_i q_{ij}\right) h_j$$

and hence, that $\lambda \in S'$. Since $\Lambda_{S'S}$ is an isometric $*$-morphism we have $\overline{\mathfrak{C}(S)} \subset \overline{\mathfrak{C}(S')}$. Hence we have, that

$$\lim_{\rightarrow} \overline{\mathfrak{C}(S)} = \overline{\bigcup_{S \in L} \mathfrak{C}(S)}$$

We now show, that $\bigcup_{S\in L} \mathfrak{C}(S) = \mathfrak{C}_x$, since then the claim follows. Therefore let first $N = \sum_{i=1}^m z_i N_{\alpha_i} \in \bigcup_{S\in L} \mathfrak{C}(S)$. Now each α_i is a map $\alpha_i : \{x\} \rightarrow S_i$ for a S_i generated by rotationally independent numbers $(k_1^{(i)}, ..., k_{n_i}^{(i)})$. I.e. $\alpha_i(x) \in S_i = S(k_1^{(i)}, ..., k_{n_i}^{(i)})$. Now we have, since L_2 is directed, that there is a $\hat{S} \geq S_1, ..., S_m$ with $S_i \subseteq \hat{S}$ for $i = 1, ..., m$ and hence $\alpha_i(x) \in \hat{S}$. With this we have $N \in \mathfrak{C}_x(\hat{S})$ and since $\mathfrak{C}_x(\hat{S}) \subset \mathfrak{C}_x$ is a subalgebra, it follows, that $N \in \mathfrak{C}_x$. Now let $N = \sum_{i=1}^m z_i N_{\alpha_i} \in \mathfrak{C}_x$. Then with the same argumentation as before we have a $\hat{S} \in L_2$ with $N_{\alpha_i} \in \mathfrak{C}_x(\hat{S})$ for all $i = 1, ..., m$. Since $\mathfrak{C}_x(\hat{S}) \subset \bigcup_{S\in\mathcal{L}} \mathfrak{C}_x(S)$, then the claim follows.

□

Hence we have expressed the full point holonomy algebra using two inductive limits and arrived at an elementary C^*-algebra, whose spectrum can be calculated by easy algebraic methods.

5.4. Spectral Theory of $\mathfrak{C}_x(S)$

In this section we want to calculate the spectrum of $\mathfrak{C}_x(S)$. Due to the elementary structure of this algebra, this can be done in a straightforward way. But before doing so we show the following:

Lemma 28 (cp. Ch. 28 of [31])
Let $S \in L$ given by $S = S(k_1, ..., k_{n_S})$. Then there is a canonical isomorphism

$$\mathrm{Hom}(S, U(1)) \cong U(1)^{n_S}$$

given by:

$$x_S \in \mathrm{Hom}(S, U(1)) \mapsto (x_S(k_i))_{i=1}^{n_S} \in U(1)^{n_S}$$

Proof. Since a group homomorphism is uniquely defined by the action on the generators of the group, an inverse for the map is easily constructed. Hence it follows, that it is bijective. □

We use this for the following definition:

Definition 33 (cp. Ch. 28 of [31])
Let $S \in L$ given by $S = S(k_1, ..., k_{n_S})$. Then define the topological space

$$X_S = \mathrm{Hom}(S, U(1))$$

topologized via the pullback topology from $U(1)^{n_S}$ using the canonical isomorphism $X_S \cong U(1)^{n_S}$.

We now can show the following, which qualifies X_S as the spectrum of an abelian, unital C^*-algebra:

Lemma 29
X_S is a compact Hausdorff space.

Proof. Follows directly, since $U(1)^{n_S}$ is a compact Hausdorff space. □

We now finally calculate the spectrum of $\mathfrak{C}_x(S)$:

Theorem 36
Let $S \in L$ given by $S = S(k_1, ..., k_{n_S})$. Then:

$$\Delta(\mathfrak{C}_x(S)) = X_S$$

Proof. We first construct the conjectured homeomorphism $\Psi : \Delta(\mathfrak{C}(S)) \to X_S$:

$$\Psi : \Delta(\mathfrak{C}(S)) \to X_S, \chi \mapsto (\Psi\chi : S \to U(1), \lambda \mapsto \chi(N_\lambda))$$

Where we have identified $S \cong \{\lambda : \Sigma \to S | \operatorname{supp}(\lambda) \subset \{x\}\}$ via the obvious identification $\lambda \mapsto (x \mapsto \lambda)$.

We now have to show, that this map is a homeomorphism. Since $\Delta(\mathfrak{C}(S))$ and X_S are compact Hausdorff spaces, it suffices to show, that this map is a well defined, continuous bijection.

To show, that it is a bijection, we construct an inverse: Let $(R : S \to U(1)) \in X_S$. Then define:

$$\Psi^{-1}R : \mathfrak{C}(S) \to \mathbb{C}, N_\lambda \mapsto R(\lambda(x))$$

This defines a $*$-morphism by lemma 38 since R is a homomorphism. We now have to show, that:

$$\begin{aligned}
&\forall \chi \in \Delta(\overline{\mathfrak{C}(S)}) : \Psi\chi \in X_S \\
&\forall R \in X_S : \Psi^{-1}R \in \Delta(\overline{\mathfrak{C}(S)}) \\
&\Psi \circ \Psi^{-1} = id. \wedge \Psi^{-1} \circ \Psi = id.
\end{aligned}$$

Then it follows, that Ψ is a well-defined bijection with inverse Ψ^{-1}.

We first show, that $\forall \chi \in \Delta(\overline{\mathfrak{C}(S)}) : \Psi\chi \in X_S$.

Therefore let $\chi \in \Delta(\overline{\mathfrak{C}(S)})$, i.e. $\chi : \overline{\mathfrak{C}(S)} \to \mathbb{C}$ is a non-zero $*$-homomorphism. We now want to show, that $\Psi\chi \in X_S$. Therefore observe first, that for $\lambda \in S$ we have:

$$\begin{aligned}
|\Psi\chi(\lambda)|^2 &= |\chi(N_\lambda)|^2 \\
&= \chi(N_\lambda \overline{\chi(N_\lambda)} \\
&= \chi(N_\lambda)\chi(N_\lambda^*) \\
&= \chi(N_\lambda)\chi(N_{-\lambda}) \\
&= \chi(N_\lambda N_{-\lambda}) \\
&= \chi(N_0) \\
&= \chi(\mathbb{1}) \\
&= 1
\end{aligned}$$

Here we have used remark 3 in the last step. Hence we have, that $\chi(N_\lambda) \in \{u \in \mathbb{C} | |u| = 1\} = U(1)$ for all $\lambda \in S$.

We now have to show, that Ψ defines a group homomorphism:

$$\begin{aligned}
\forall \lambda_1, \lambda_2 \in S : (\Psi\chi)(\lambda_1 + \lambda_2) &= \chi(N_{\lambda_1+\lambda_2}) \\
&= \chi(N_{\lambda_1} N_{\lambda_2} \\
&= \chi(N_{\lambda_1})\chi(N_{\lambda_2}) \\
&= (\Psi\chi)(\lambda_1)(\Psi\chi)(\lambda_2)
\end{aligned}$$

Hence it is a homomorphism.

We now show that $\Psi^{-1}R \in \Delta(\overline{\mathfrak{C}(S)})$ holds for all $R \in X_S$.

Let $R \in X_S$, i.e. $R : S \to U(1)$ is homomorphism. Now we have to show, that $\Psi^{-1}R$ is a non-zero $*$-morphism. By lemma 38 it remains to show, that $\Psi^{-1}R(N_\alpha N_\beta) = \Psi^{-1}R(N_\alpha)\Psi^{-1}R(N_\beta)$ and $\Psi^{-1}R(N_\alpha^*) = \overline{\Psi^{-1}R(N_\alpha)}$ hold. But those follow directly:

$$\begin{aligned}(\Psi^{-1}R)(N_\alpha N_\beta) &= (\Psi^{-1}R)(N_{\alpha+\beta}) \\ &= R(\alpha+\beta) \\ &= R(\alpha)R(\beta) \\ &= (\Psi^{-1}R)(N_\alpha)(\Psi^{-1}R)(N_\beta)\end{aligned}$$

and

$$\begin{aligned}(\Psi^{-1}R)(N_\alpha^*) &= (\Psi^{-1}R)(N_{-\alpha}) \\ &= R(-\alpha) \\ &= \overline{R(\alpha)} \\ &= \overline{\Psi^{-1}R(N_\alpha)}\end{aligned}$$

And hence $\Psi^{-1}R$ is a $*$-morphism by lemma 38. Further it holds, that $\Psi^{-1}R$ is non-zero, since $R(0) = 1$ and hence

$$\Psi^{-1}R(N_0) = R(0) = 1$$

for any $R \in X_S$.

We now want to show, that Ψ^{-1} is really the inverse of Ψ.

Therefore let first $\chi \in \Delta(\overline{\mathfrak{C}(S)})$. Then $\Psi\chi(\lambda) = \chi(N_\lambda)$ holds for $\lambda \in S$. And hence we have:

$$\Psi^{-1}(\Psi\chi)(N_\lambda) = \chi(N_\lambda) \Rightarrow \Psi^{-1}\Psi\chi = \chi \Rightarrow \Psi^{-1} \circ \Psi = id.$$

On the other hand $\Psi^{-1}R(N_\lambda) = R(\lambda)$ holds for $R \in X_S$. And further $\Psi(\Psi^{-1}R)(\lambda) = \Psi^{-1}R(N_\lambda) = R(\lambda)$. Hence we have $\Psi \circ \Psi^{-1} = id.$ and $\Psi^{-1} \circ \Psi = id.$. Hence bijectivity is shown.

Now we have to show continuity. Therefore let $(\chi_n)_{n\in I}$ be a net in $\Delta(\overline{\mathfrak{C}}(S))$ with $\lim \chi_n = \chi \in \Delta(\overline{\mathfrak{C}}(S))$. I.e.:

$$\forall x \in \overline{\mathfrak{C}(S)} : \forall \epsilon > 0 : \exists N \in I : \forall n \geq N : |\chi_n(x) - \chi(x)| < \epsilon$$

Now let $(k_1, ..., k_{n_S})$ be a set of rationally independent numbers generating S. Then above gives:

$$\forall i = 1, ..., n_S : \forall \epsilon > 0 : \exists N \in I : \forall n \geq N : |\chi_n(N_{k_i}) - \chi(N_{k_i})| < \epsilon$$

This can be written as:

$$\forall i = 1, ..., n_S : \forall \epsilon > 0 : \exists N \in I : \forall n \geq N : |(\Psi\chi_n)(k_i) - (\Psi\chi)(k_i)| < \epsilon$$

Since the topology on X_S is defined via the identification $X_S \cong U(1)^{n_S}$, this implies, that

$$\lim(\Psi\chi_n) = \Psi\chi$$

and hence continuity is shown. □

5.5. Spectral Theory of the Point Holonomy Algebra

By theorem 18 we know, that we now have to calculate the projective limits of the spectra to arrive at the spectrum of the point holonomy algebra. Since we have two nested inductive limits on the C^*-side, we also have two nested projective limits on the topological side.

5.5.1. Spectral Theory of the Algebra of Almost Periodic Functions

We will first define the Bohr compactification of the real line $\bar{\mathbb{R}}_{Bohr}$ as the projective limit of the X_S. By theorem 18 this makes $\bar{\mathbb{R}}_{Bohr}$ to the spectrum of $\overline{\mathfrak{C}}_x$ by definition.

Definition 34 (Bohr compactification of the real line, cp. [31], [34]) *1. Define a projective family of topological spaces $(X_S, p_{SS'})_{S,S'\in L_2}$ via X_S defined as before and:*

$$p_{SS'} : X_{S'} \to X_S, x_{S'} \mapsto (x_{S'})|_S$$

2. Define the Bohr compactification of the real line as

$$\overline{\mathbb{R}}_{Bohr} = \lim_{\leftarrow} X_S$$

We now show, that this is well defined:

Theorem 37
$(X_S, p_{SS'})_{S,S'\in L_2}$ is a projective family of compact Hausdorff spaces.

Proof. That X_S is a compact Hausdorff spaces for each $S \in L_2$ was already shown in lemma 29. Further $p_{SS'}$ satisfies the commutativity properties, since it maps a homomorphism to a restriction, which satisfies this requirements trivially. Further it is continuous, since it maps $X'_S \cong U(1)^{n'_S}$ on its subspsace $U(1)^{n_S}$. □

Hence we obtain directly by theorem 18 and definition 34:

Corollar 7
Let anything be as before. Then:

$$\Delta(\overline{\mathfrak{C}}_x) = \overline{\mathbb{R}}_{Bohr}$$

Proof. By theorem 18 and definition 34 we have directly:

$$\Delta(\overline{\mathfrak{C}_x}) = \Delta(\lim_{\rightarrow} \overline{\mathfrak{C}(S)}) = \lim_{\leftarrow} \Delta(\overline{\mathfrak{C}(S)}) = \lim_{\leftarrow} X_S = \overline{\mathbb{R}}_{Bohr}$$

□

5.5.2. Spectral Theory of the Point Holonomy Algebra

We now want to calculate the spectrum of the full point holonomy algebra. This is done in the following:

Theorem 38
Let anything be as before. Then:

1. $\Delta(\overline{\mathfrak{C}_V}) = \mathcal{A}_V$ *where* $\mathcal{A}_V = \{\phi : V \to \overline{\mathbb{R}}_{Bohr}\} \cong \prod_{v\in V} \bar{\mathbb{R}}_{Bohr}$.

2. $\Delta(\overline{\mathfrak{C}}) = \mathcal{A}$ *where* $\mathcal{A} = \{\phi : \Sigma \to \overline{\mathbb{R}}_{Bohr}\} \prod_{v \in \Sigma} \overline{\mathbb{R}}_{Bohr}$.

Proof. 1.) We show first the prototypical case:

$$\Delta(\overline{\mathfrak{C}}_{\{x,y\}}) = \Delta(\overline{\mathfrak{C}}_{\{x\}}) \times \Delta(\overline{\mathfrak{C}}_{\{y\}})$$

Therefore recall first from theorem 35, that $\overline{\mathfrak{C}}_{\{x\}}$ and $\overline{\mathfrak{C}}_{\{y\}}$ can be seen as subalgebras of $\overline{\mathfrak{C}}$ since the maps

$$i_x : N_\alpha \in \mathfrak{C}_{\{x\}} \mapsto N_\alpha \in \overline{\mathfrak{C}}_{\{x,y\}}$$
$$i_y : N_\beta \in \mathfrak{C}_{\{x\}} \mapsto N_\beta \in \overline{\mathfrak{C}}_{\{x,y\}}$$

extend to isometric $*$-morphism and hence are especially injective.

We now define the conjectured homeomorphism

$$\Psi : \Delta(\overline{\mathfrak{C}}_{\{x,y\}}) \to \Delta(\overline{\mathfrak{C}}_{\{x\}}) \times \Delta(\overline{\mathfrak{C}}_{\{y\}})$$

by:

$$\chi \mapsto \Psi(\chi) = (\chi \circ i_x, \chi \circ i_y) = (i_x^* \chi, i_y^* \chi)$$

That i_x^* and i_y^* define indeed nonzero $*$-morphisms was shown in lemma 13.

We now construct the inverse:

$$\Psi^{-1} : \Delta(\overline{\mathfrak{C}}_{\{x\}}) \times \Delta(\overline{\mathfrak{C}}_{\{y\}}) \to \Delta(\overline{\mathfrak{C}}_{\{x,y\}}), (\chi_x, \chi_y) \mapsto \chi$$

Here we define χ via its action on basis elements as follows: Let $\alpha : \Sigma \to \mathbb{R}$ with $\operatorname{supp}(\alpha) \subset \{x, y\}$. Now define the maps

$$\alpha_x : \Sigma \to \mathbb{R}, v \mapsto \begin{cases} \alpha(v) & \text{if } v = x \\ 0 & \text{else} \end{cases}$$

and:

$$\alpha_y : \Sigma \to \mathbb{R}, v \mapsto \begin{cases} \alpha(v) & \text{if } v = y \\ 0 & \text{else} \end{cases}$$

Observe further, that $\alpha = \alpha_x + \alpha_y$. Then define χ for $\alpha : \Sigma \to \mathbb{R}$ with $\operatorname{supp}(\alpha) \subset \{x, y\}$ as:

$$\chi\,(N_\alpha) := \chi_x(N_{\alpha_x})\chi_y(N_{\alpha_y})$$

We first show, that this χ defines really a non-zero $*$-morphism. Therefore observe, that automatically $\chi(\mathbb{1}) = \chi(N_0) = 1$ holds, and hence it is non-zero. We now use again lemma 38. Therefore let $\alpha, \beta : \Sigma \to \mathbb{R}$ with $\operatorname{supp}(\alpha) \subset \{x, y\}$ and $\operatorname{supp}(\beta) \subset \{x, y\}$. Now observe

$$\begin{aligned}
\chi(N_\alpha)\chi(N_\beta) &= \chi_x(N_{\alpha_x})\chi_y(N_{\alpha_y})\chi_x(N_{\beta_x})\chi_y(N_{\beta_y}) \\
&= \chi_x(N_{\alpha_x+\beta_x})\chi_y(N_{\alpha_y+\beta_y}) \\
&= \chi(N_{\alpha+\beta}) \\
&= \chi(N_\alpha N_\beta)
\end{aligned}$$

where we have used, that $\mathbb{C}$ is commutative and further, that $\alpha(v) + \beta(v) = (\alpha + \beta)(v)$ holds for all $v \in \Sigma$. Further observe

$$\begin{aligned}
\chi(N_\alpha^*) &= \chi(N_{-\alpha}) \\
&= \chi_x(N_{(-\alpha)_x}\chi_y(N_{(-\alpha)_y}) \\
&= \chi_x(N_{-\alpha_x})\chi_y(N_{-\alpha_y}) \\
&= \overline{\left(\chi_x(N_{\alpha_x})\chi_y(N_{\alpha_y})\right)} \\
&= \overline{\chi(N_\alpha)}
\end{aligned}$$

where we have also used, that $\mathbb{C}$ is commutative and further, that $-\alpha(v) = (-\alpha)(v)$ for all $v \in V$. And hence it extends to an unique $*$-morphism by lemma 38.

We now show, that Ψ^{-1} is really an inverse. Therefore let $\alpha : \Sigma \to \mathbb{R}$ with $\operatorname{supp}(\alpha) \subset \{x, y\}$. Now observe:

$$\chi(N_{\alpha_x})\chi(N_{\alpha_y}) = \chi(N_{\alpha_x+\alpha_y}) = \chi(N_\alpha)$$

And hence $\Psi^{-1} \circ \Psi = id.$ on basis elements. It remains to show, that $\Psi \circ \Psi^{-1} = id.$. To see this, observe first, that for $\alpha_x : \Sigma \to \mathbb{R}$ with $\operatorname{supp}(\alpha_x) \subset \{x\}$ it follows, that $(\alpha_x)_y = 0$ and hence $N_{(\alpha_x)_y} = \mathbb{1}$. Now let $(\chi_x, \chi_y) \in \Delta(\overline{\mathfrak{C}}_{\{x\}}) \times \Delta(\overline{\mathfrak{C}}_{\{y\}})$. Then $\tilde{\chi} := (\Psi(\chi_x, \chi_y))\,(N_\alpha) = \chi_x(N_{\alpha_x})\chi_y(N_{\alpha_y})$. Now $\Psi\tilde{\chi} = (\tilde{\chi} \circ i_x, \tilde{\chi} \circ i_y)$ and further for $\alpha_x : \Sigma \to \mathbb{R}$ with $\operatorname{supp}(\alpha_x) \subset \{x\}$:

$$\tilde{\chi} \circ i_x(N_{\alpha_x}) = \chi_x(N_{\alpha_x})\mathbb{1}$$

Further the analogous statement holds for an $\alpha_y : \Sigma \to \mathbb{R}$ with $\operatorname{supp}(\alpha_y) \subset \{y\}$. Hence $\Psi \circ \Psi^{-1} = id.$

Finally we have to show, that Ψ is continuous. Therefore recall first, that Ψ is continuous if and only if $\pi_x \circ \Psi$ and $\pi_y \circ \Psi$ are continuous, where π_x and π_y are the canonical projections from $\Delta(\overline{\mathfrak{C}}_{\{x\}}) \times \Delta(\overline{\mathfrak{C}}_{\{y\}})$ on its factors. Now let $(\chi_i)_{i\in I} \subset \Delta(\overline{\mathfrak{C}}_{\{x,y\}})$ be a net with $\lim \chi_i \in \Delta(\overline{\mathfrak{C}}_{\{x,y\}})$. This means:

$$\forall a \in \overline{\mathfrak{C}}_{\{x,y\}} : \forall \epsilon > 0 : \exists N \in I : \forall i \geq N : |\chi_i(a) - \chi(a)| < \epsilon$$

But since $i_x : \overline{\mathfrak{C}}_{\{x\}} \hookrightarrow \overline{\mathfrak{C}}_{\{x,y\}}$ is an embedding, this implies:

$$\forall a \in \overline{\mathfrak{C}}_{\{x\}} : \forall \epsilon > 0 : \exists N \in I : \forall i \geq N : |\chi_i \circ i_x(a) - \chi \circ i_x(a)| < \epsilon$$

and hence Ψ is continuous.

By the straightforward generalization to finite products we hence obtain:

$$\Delta(\overline{\mathfrak{C}}_V) = \prod_{v\in V} \overline{\mathbb{R}}_{Bohr} \cong \{\phi : V \to \overline{\mathbb{R}}_{Bohr}\}$$

2.) This follows with Ex. 1.1.14 of [27], where it is shown, that infinite topological products can be expressed as a projective limit over products over finite subsets of the full index set. □

We now apply our spectral theorem on $\overline{\mathfrak{C}} = \lim_{\to} \mathfrak{C}_V$.

5.6. Discussion

In this section the relevance of the concepts developed in part I were illustrated by applying them to the calculation of the quantum configuration space of polymer matter. The quantum configuration space obtained with our methods equals the result presented in the literature (see e.g. [7]). Further the concepts of part I give some further result: On the one hand it follows from theorem 27 that each cyclic representation of the point holonomy algebra arises as an inductive limit of representations of the corresponding inductive families. Further it follows with theorem 14, that an orthonormal basis for the inductive limit representation is determined by the orthonormal bases of the corresponding members of the inductive family.

6. Application to Loop Quantization of Gravity

The aim of this chapter is the calculation of the quantum configuration space of loop quantum gravity using the methods developed in this thesis. Therefore the first section provides a very condensed introduction to the configuration spaces of $SU(2)$ gauge theories. The second section introduces holonomies and Wilson lines, which constitute the classical configuration variables on which the quantum theory of loop quantum gravity is based. The third section introduces the groupoid of paths as well as the hoop group, which are both important concepts for defining the quantum algebra and investigating its inductive limit decomposition. In the next section the quantum algebra of loop quantum gravity is defined and it is shown, that it defines an abelian, unital C^*-algebra. The fifth section expresses the quantum algebra of loop quantum gravity as an inductive limit of more elementary C^*-algebras. In the sixth section dually the corresponding projective family of topological spaces together with a compatible family of group actions is presented. The seventh section investigates in which sense the Wilson algebra arises as the fixed point algebra of a larger C^*-algebra. Further by this and earlier results of this chapter the quantum configuration space of loop quantum gravity is computed using the methods developed in chapter 3 and chapter 4.

The content of the first chapter can be found in any basic reference on mathematical gauge theory (e.g. [8] or [16]). Further the formulation of gravity as a $SU(2)$ gauge theory is not presented here and the author encourages the reader of this thesis to consult basic loop quantum gravity literature for this topic (e.g. Ch. 4 of [31]). The definitions used in Section 2, 3 and 4 are oriented themselves at [3] and [4]. The topological constructions appearing in Section 6 are standard in loop quantum gravity and our reference for this topic is given by Ch. 6.2 of [31]. The corresponding inductive limit construction on the C^*-side given in section 5 is firstly presented in this thesis. Further the application of the results of chapter 3 and chapter 4 to the calculation of the quantum configuration space of loop quantum gravity - as it happens in section 7 - is also firstly presented in this thesis.

The author wants to emphasize, that the proofs in this chapter, which are not connected with the main results of this thesis, are often only sketched or ommited, if they appear in some reference. Otherwise this chapter would have got very long.

6.1. Crash Course in $SU(2)$-Principal Fiber Bundles

Let Σ be a compact, orientable, 3-dimensional, smooth, Riemannian manifold. Let further $(P, \pi, \Sigma; SU(2))$ be a $SU(2)$-principal fiber bundle and $(E, \pi, \Sigma, \mathfrak{su}(2))$ be a $SU(2)$-vector bundle with typical fiber $\mathfrak{su}(2)$ associated to P via Ad. Further we abbreviate in this section Principal Fiber Bundle by PFB and Vector Bundle by VB.

We first recall an important theorem:

Theorem 39 (cp. [25], [31], [8])
Let Σ be a compact, orientable 3-manifold. Then:

1. *Each $SU(2)$-PFB over Σ is trivial.*
2. *Each VB associated to a trivial PFB possesses a global frame.*
3. *$T\Sigma$ is parallelizable.*

Proof. 1.) See [25] and [31]. 2.) See [8]. 3.) See [25] and [31]. □

Now choose a global trivialization of P and a corresponding global frame of E. Hence any Ehresmann connection corresponds in those coordinates to a global connection 1-form $A \in \Omega^1(\Sigma, \mathfrak{su}(2))$ (cp. [8]). We now want to study the transformation properties of this global connection 1-form. Therefore recall (cp. [8]), that for trivial PFBs a global frame is given by a global section $\xi \in \Gamma(P)$ and a change of the global trivialization is given by a map $g \in C^\infty(\Sigma, G)$ via $\xi \mapsto g\xi$. Further this induces the following transformation on the global connection 1-form (cp. [8] and [31] for this convention):

$$A \mapsto gAg^{-1} - dgg^{-1}$$

In the following we set

$$\begin{aligned} \mathcal{A}_{class} &:= \Omega^1(\Sigma, \mathfrak{su}(2)) \\ \mathcal{G} &:= C^\infty(\Sigma, SU(2)) \end{aligned}$$

and write above group-action as:

$$\mathcal{G} \curvearrowright \mathcal{A}$$

This can be used to define $\tilde{A}_{class} = \mathcal{A}/\mathcal{G}$. A theory whose configuration space is given by $\tilde{A}_{class}$ will be called a $SU(2)$ gauge theory. In this case $\mathcal{A}_{class}$ corresponds to some kind of "overcomplete" configuration space whose redundancy is modelled an action of the group $\mathcal{G}$. It is the basis of loop quantum gravity, that gravity can be modelled as a $SU(2)$ gauge theory. The basic idea behind this formulation of gravity is to introduce a bundle morphism $e : T\Sigma \to E$ and define a Ehresmann connection on E using the Levi Civita connection on $T\Sigma$. By smart canonical transformations on this new gravitational phase space it is not only possible to mimic the kinematical structure of a $SU(2)$ gauge theory, but also to obtain a constraint structure that is similar to Yang Mills theories with gauge group $SU(2)$.

6.2. Classical Configuration Variables: Holonomies and Wilson Lines

Before defining the new classical configuration variables used for loop quantization of gravity, we need some preliminary notions:

Definition 35 (cp. [31])
Let Σ be a compact, orientable, smooth 3-manifold.

1. *Let $\mathcal{C}$ be the space of all continuous, oriented, piecewise analytic, parametrized curves in Σ, i.e. $c \in \mathcal{C}$ if and only if*
   a) $c : [0, t_1] \cup [t_1, t_2] \cup ... \cup [t_{n-1}, t_n] \to \Sigma$ and

b) c is continuous and

c) $c|_{[t_k,t_{k+1}]}$ is analytic and

d) $c((t_k, t_{k+1}))$ is submanifold.

2. *Define maps $b, f : \mathcal{C} \to \Sigma$ via*

$$\begin{aligned} b : c &\mapsto c(0) \\ f : c &\mapsto c(1) \end{aligned}$$

3. *Define the maps*

$$\begin{aligned} \circ : \{(c_1, c_2) \in \mathcal{C} \times \mathcal{C} | f(c_1) = b(c_2)\} &\mapsto \mathcal{C} \\ \cdot^{-1} : \mathcal{C} &\to \mathcal{C} \end{aligned}$$

via

$$\circ : (c_1, c_2) \mapsto \left(c_1 \circ c_2 : t \mapsto \left\{ \begin{array}{cc} c_1(2t) & t \in [0, 1/2) \\ c_2(2t-1) & t \in [1/2, 1] \end{array} \right. \right)$$

and

$$\cdot^{-1} : c \mapsto \left(c^{-1} : t \mapsto c(1-t)\right)$$

4. *A retracing is a curve $d \in \mathcal{C}$ such that there is a curve $c \in \mathcal{C}$ with $d = c \circ c^{-1}$.*

5. *Let $x_0 \in \Sigma$. Define the space of loops based at x_0 as*

$$\mathcal{C}_{x_0} = \{c \in \mathcal{C} | b(c) = f(c) = x_0\}$$

With this we define holonomies and Wilson lines as observables on $\mathcal{A}$:

Definition 36
(cp. [31]) Let anything be as in the last definition. Let $e_{SU(2)}$ denote the unit element of $SU(2)$. Then:

1. *The holonomy $H : \mathcal{C} \times \mathcal{A} \to SU(2), (c, A) \mapsto H_c[A]$ is defined as the solution of the following differential equation:*

$$\begin{aligned} \dot{H}_c[A](t) &= H_c[A](t) A_{c(t)} \dot{c}(t) \\ H_c[A](0) &= e_{SU(2)} \\ H_c[A] &:= H_c[A](1) \end{aligned}$$

2. *The Wilson line is defined as*

$$T : \mathcal{C} \times \mathcal{A} \to \mathbb{C}, (c, A) \mapsto T_c[A] := \frac{1}{2} \mathrm{tr}\left(H_c[A]\right)$$

Further we call T_c a Wilson loop if $c \in \mathcal{C}_{x_0}$ for some $x_0 \in \Sigma$.

We first have the following elementary statemant, whose proof can be found in any elementary reference for holonomy theory:

Lemma 30 (cp. p. 165 of [31])
Let c, c_1, c_2 be curves. Let further $f(c_1) = b(c_2)$. Then we have for all $A \in \mathcal{A}$:

$$H_{c_1 \circ c_2}[A] = H_{c_1}[A] H_{c_2}[A]$$
$$H_{c^{-1}}[A] = H_c^{[}A]$$

We now want to investigate the behavior of holonomies and Wilson lines under gauge transformations:

Proposition 2 (cp. p. 165 of [31])
Let $\mathcal{G} \curvearrowright \mathcal{A} : (g, A) \mapsto gA$ be the group action defined in section 6.1. Let $g \in \mathcal{G}$, $A \in \mathcal{A}$ and $c \in \mathcal{C}$. Let $x_0 \in \Sigma$. Then:

1. $H_c[gA] = g(b(c)) H_c[A] g(f(c))^{-1}$

2. *For $c \in \mathcal{C}_{x_0}$:* $T_c[gA] = T_c[A]$

Proof. 1.) See p. 165 of [31]
2.) Follows directly by $b(c) = f(c)$ and cyclicity of the trace. □

Further we want to investigate the codomain of the Wilson loop:

Proposition 3
Let $c \in \mathcal{C}_{x_0}$ and $A \in \mathcal{A}$. Then:

$$T_c[A] \in [-1, +1] \subset \mathbb{R}$$

Proof. Recall, that (cp. Ex. 1.5 of [15]):

$$SU(2) = \left\{ \left(\begin{array}{cc} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{array} \right) \middle| \alpha, \beta \in \mathbb{C} \wedge |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Hence for all $U \in SU(2)$ we have:

$$\mathrm{tr}(U) = \alpha + \bar{\alpha} = 2Re(\alpha)$$

Now we have $|\alpha|^2 = 1 - |\beta|^2 \leq 1$ and with this $\mathrm{Re}(\alpha)^2 + \mathrm{Im}(\alpha)^2 \leq 1$. This gives $|\mathrm{Re}(\alpha)| \leq 1$, which implies:

$$\forall U \in SU(2) : \left| \frac{1}{2} \mathrm{tr}(U) \right| \leq 1$$

And hence the claim follows. □

We now define Wilson Loops on $\mathcal{A}$:

Definition 37 (cp. [3])
Let $x_0 \in \Sigma$ be arbitrary but fixed. Then the Wilson loop is defined as:

$$T : \mathcal{C}_{x_0} \times \mathcal{A} \to [-1, 1], (c, [A]) \mapsto T_c[A]$$

6.3. The Groupoid of Paths and the Hoop Group

We first define the notion of a groupoid:

Definition 38 (cp. Def. 6.2.6 of [31])
A groupoid is a small category in which every morphism is iso.

We have already understood, that we can compose paths in $\mathcal{C}$ and also have some notion of an inverse path. Unfortunately this gives not a groupoid, since the composition of a path with its inverse path gives a retracing. In order to upgrade the set of paths to a groupoid, we have to introduce an equivalence relation on $\mathcal{C}$:

Definition 39 (cp. Def. 6.2.3 of [31])
Define the following equivalence relation on $\mathcal{C}$:
$\forall c_1, c_2 \in \mathcal{C} : c_1 \sim c_2 \Leftrightarrow$:

1. $b(c_1) = b(c_2) \wedge f(c_1) = f(c_2)$
2. *c_1 is equivalent to c_2 up to a finite number of retracings and an analytic reparametrization.*

Further we set

$$\mathcal{P} = \mathcal{C}/\sim$$

We have then the following theorem:

Theorem 40 (cp. p. 164 and p. 167 of [31])
Let anything be as before. Then:

1. *$\sim$ is an equivalence relation.*
2. *$\mathcal{P}$ is a groupoid with $\mathrm{obj}(\mathcal{P}) = \Sigma$, and $\mathrm{mor}(x, y) = \mathcal{C}(x, y)/\sim$, where $\mathcal{C}(x, y)$ denotes the set of paths from x to y.*
3. *Let G be a groupoid and $x \in \mathrm{obj}(G)$. Then the vertex group $\mathrm{mor}(x, x)$ is a group.*

Proof. See p. 164 and p. 167 of [31]. □

With this we define:

Definition 40 (cp. p. 167 of [31])
Let $x_0 \in \mathrm{obj}(\mathcal{P}) = \Sigma$. Then define the hoop group $\mathcal{H}_{x_0}$ as:

$$\mathcal{Q}_{x_0} = \mathrm{mor}(x_0, x_0)$$

6.4. The Wilson Algebra

In the section 6.2 we introduced certain configuration variables on the space $\mathcal{A}_{class}$ of Ehresmann connections. In this section we want to define the corresponding quantum algebra of observables, where we will follow [3]. Therefore we first want to investigate, if holonomies and Wilson loops are not only well defined on $\mathcal{C}_{x_0}$, but also on the hoop group:

Proposition 4 (cp. p. 165 of [31])
The holonomy and the Wilson loop are well defined maps on $\mathcal{Q}_{x_0}$. I.e. for $\alpha \in \mathcal{Q}_{x_0}$ the following holds:

$$\forall c_1, c_2 \in \alpha : \forall A \in \mathcal{A} : H_{c_1}[A] = H_{c_2}[A]$$
$$\forall c_1, c_2 \in \alpha : \forall A \in \mathcal{A} : T_{c_1}[A] = T_{c_2}[A]$$

Proof. It is explained on p. 165 of [31], that the holonomy is well defined on $\mathcal{Q}_{x_0}$. Since the Wilson Loop is just the trace over a holonomy, the claim follows also for the Wilson loop. □

In the following we will consider H and T as maps on $\mathcal{Q}_{x_0}$, i.e.:

$$H : \mathcal{Q}_{x_0} \times \mathcal{A} \mapsto SU(2), (\alpha = [c], A) \mapsto H_\alpha[A] := H_c[A]$$
$$T : \mathcal{Q}_{x_0} \times \mathcal{A} \mapsto SU(2), (\alpha = [c], A) \mapsto T_\alpha[A] := T_c[A]$$

We now want to show, that the holonomy can be regarded as a group homomorphism mapping the hoop group on $SU(2)$:

Lemma 31
Let $A \in \mathcal{A}$. Then the map $H[A] : \mathcal{Q}_{x_0} \to SU(2), \alpha \mapsto H_\alpha[A]$ is a group homomorphism.

Proof. This follows directly with lemma 30 and proposition 4. □

We now want to investigate, which relations the Wilson loop satisfies:

Proposition 5
Let anything be as before. Then:

1. $\forall \alpha, \beta \in \mathcal{Q}_{x_0} : T_\alpha T_\beta = \frac{1}{2}(T_{\alpha\circ\beta} + T_{\alpha\circ\beta^{-1}})$.
2. $\forall \alpha \in \mathcal{Q}_{x_0} : T_{\alpha^{-1}} = T_\alpha$.
3. $\forall \alpha, \beta \in \mathcal{Q}_{x_0} : T_{\alpha\circ\beta} = T_{\beta\circ\alpha}$.

Proof. 1.) + 2.) Please recall first the famous Mandelstam identities (cp. [24]): Let $M_1, M_2 \in SU(2)$. Then:

$$\operatorname{tr}(M_1)\operatorname{tr}(M_2) = tr(M_1 M_2) + \operatorname{tr}(M_1 M_2^{-1})$$
$$\operatorname{tr}(M_1) = \operatorname{tr}(M_1^{-1})$$

With this, the assertions 1.) and 2.) follows directly.
3.) Using the cyclicity of the trace and lemma 31, we have directly the following:

$$\begin{aligned} T_{\alpha\circ\beta} &= \operatorname{tr}(H_{\alpha\circ\beta}) \\ &= \operatorname{tr}(H_\alpha H_\beta) \\ &= \operatorname{tr}(H_\beta H_\alpha) \\ &= \operatorname{tr}(H_{\beta\circ\alpha}) \\ &= T_{\beta\circ\alpha} \end{aligned}$$

and hence the claim follows. □

We now describe the construction of the quantum Wilson algebra. Let $F\mathcal{C}_{x_0}$ be the free complex vector space over $\mathcal{C}_{x_0}$. We write in the following the basis element corresponding to $\alpha \in \mathcal{C}_{x_0}$ as T_{x_0} in slight abuse of notation. It will be always clear in the following, if this denotes an element of $F\mathcal{C}_{x_0}$ or the corresponding classical observable.

We now define a subspace $K \subset F\mathcal{C}_{x_0}$, which will be a 2-sided ideal:

$$K = \left\{ \sum_{i=1}^{n} z_i T_{\alpha_i} \in F\mathcal{C}_{x_0} \middle| \forall A \in \mathcal{A} : \sum_{i=1}^{n} z_i T_\alpha[A] = 0 \right\}$$

Here the first T_{α_i} denote the corresponding elements of $F\mathcal{C}_{x_0}$ and the latter T_{α_i} denote the Wilson loop on $\mathcal{A}$. We then define a product on $F\mathcal{C}_{x_0}$ as

$$\cdot : F\mathcal{C}_{x_0} \times F\mathcal{C}_{x_0} \to F\mathcal{C}_{x_0}, \left(\sum_{i=1}^{n} z_i T_{\alpha_i} \right) \cdot \left(\sum_{j=1}^{m} \tilde{z}_j T_{\beta_j} \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} z_i \tilde{z}_j T_{\alpha_i} T_{\beta_j}$$

where:

$$T_{\alpha_i} T_{\beta_j} := \frac{1}{2} \left(T_{\alpha_i \circ \beta_j} + T_{\alpha_i \circ \beta_j^{-1}} \right)$$

We further define an involution on $F\mathcal{C}_{x_0}$ via:

$$\left(\sum_{i=1}^{n} z_i T_{\alpha_i} \right)^* = \sum_{i=1}^{n} \bar{z}_i T_{\alpha_i}$$

Finally a norm is defined on $F\mathcal{C}_{x_0}$ via:

$$\left\| \sum_{i=1}^{n} z_i T_{\alpha_i} \right\| := \sup_{A \in \mathcal{A}} \left| \sum_{i=1}^{n} z_i T_{\alpha_i}[A] \right|$$

All together this gives:

Definition 41 (cp. [3])
The Wilson algebra $\overline{\mathcal{W}}_{x_0}$ *is defined as* $\mathcal{W}_{x_0} = F\mathcal{C}_{x_0}/K$ *together with above* $*$*-algebra structure and above norm.*

We now show the following:

Lemma 32
Let anything be as above. Then:

1. K *is a two-sided ideal.*
2. $\| \cdot \|$ *is submultiplicative, satisfies the* C^**-property and is independent of the representative.*
3. $\overline{\mathcal{W}}_{x_0}$ *is abelian and unital.*
4. *The multiplication in* $\overline{\mathcal{W}}_{x_0}$.

Hence $\mathcal{W}_{x_0}$ *is an abelian, unital* C^**-algebra.*

Proof. 1.) We show, that K is a right ideal. That it is a left ideal follows analogously. Therefore let $x = \sum_{i=1}^{n} z_i T_{\alpha_i} \in K$ and $y = \sum_{j=1}^{m} \tilde{z}_j T_{\alpha_j} \in F\mathcal{C}_{x_0}$. Now:

$$\left(\sum_{i=1}^{n} z_i T_{\alpha_i} \sum_{j=1}^{m} \tilde{z}_j T_{\alpha_j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} z_i \tilde{z}_j \frac{1}{2}\left(T_{\alpha_i \circ \alpha_j} + T_{\alpha_i \circ \alpha_j^{-1}}\right)$$

Now the Mandelstam identities give:

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{m} z_i \tilde{z}_j \frac{1}{2}\left(T_{\alpha_i \circ \alpha_j} + T_{\alpha_i \circ \alpha_j}\right)\right)[A] = \left(\sum_{i=1}^{n} z_i T_{\alpha_i}[A]\right)\left(\sum_{j=1}^{m} \tilde{z}_J T_{\alpha_j}[A]\right) = 0$$

and hence $xy \in K$.
2.) That the norm is submultiplicative follows easily as follows: Let $N_1, N_2 \in F\mathcal{C}_{x_0}$. Then:

$$\begin{aligned}
\|N_1 N_2\| &= \sup_{A \in \mathcal{A}} |N_1[A] N_2[A]| \\
&\leq \sup_{A \in \mathcal{A}} \left(\sup_{A \in \mathcal{A}} |N_1[A]|\right) |N_2[A]| \\
&= \|N_1\| \|N_2\|
\end{aligned}$$

That it satisfies the C^*-property follows also directly. Therefore let $N = \sum_{i=1}^{n} z_i T_{\alpha_i} \in F\mathcal{C}_{x_0}$. Then:

$$\begin{aligned}
\|N^* N\| &= \sup_{A \in \mathcal{A}} \left| \left(\sum_{i,j=1}^{n} \bar{z}_i z_i T_{\alpha_i} T_{\alpha_j}\right)[A] \right| \\
&= \sup_{A \in \mathcal{A}} \left|\sum_{i=1}^{n} z_i T_{\alpha 1}[A]\right| \left|\sum_{i=1}^{n} z_i T_{\alpha 1}[A]\right| \\
&= \sup_{A \in \mathcal{A}} \left|\sum_{i=1}^{n} z_i T_{\alpha 1}[A]\right|^2 \\
&= \left(\sup_{A \in \mathcal{A}} \left|\sum_{i=1}^{n} z_i T_{\alpha 1}[A]\right|\right)^2 \\
&= \|N\|^2
\end{aligned}$$

Finally we have to show, that the norm is independent of the representant of the equivalence classes of $F\mathcal{C}_{x_0}/K$. But this follows directly, since $\forall x \in K$ we have, that $x[A] = 0$.
3.) That it $\overline{\mathcal{W}}_{x_0}$ is abelian follows by $T_{\alpha^{-1}} = T_\alpha$ and the cyclicity of the trace: Therefore observe first, that in $F\mathcal{C}_{x_0}$ the following holds:

$$\begin{aligned}
T_\alpha T_\beta - T_\beta T_\alpha &= \frac{1}{2}\left(T_{\alpha \circ \beta} + T_{\alpha \circ \beta^{-1}} - T_{\beta \circ \alpha} - T_{\beta \circ \alpha^{-1}}\right) \\
&= \frac{1}{2}\left(T_{\alpha \circ \beta} - T_{\beta \circ \alpha}\right) + \frac{1}{2}\left(T_{\alpha \circ \beta^{-1}} - T_{\beta \circ \alpha^{-1}}\right)
\end{aligned}$$

Further for any $A \in \mathcal{A}$ the following holds:

$$\begin{aligned}
T_{\alpha \circ \beta}[A] &= \mathrm{tr}\left(H_{\alpha \circ \beta}[A]\right) \\
&= \mathrm{tr}\left(H_\alpha[A] H_\beta[A]\right) \\
&= \mathrm{tr}\left(H_\beta[A] H_\alpha[A]\right) \\
&= \mathrm{tr}\left(H_{\beta \circ \alpha}[A]\right) \\
&= T_{\beta \circ \alpha}[A]
\end{aligned}$$

Finally we have for any $A \in \mathcal{A}$:

$$\begin{aligned} T_{\alpha\circ\beta^{-1}}[A] &= T_{(\alpha\circ\beta^{-1})^{-1}}[A] \\ &= T_{\beta\circ\alpha^{-1}}[A] \end{aligned}$$

And hence we have:

$$[T_\alpha, T_\beta] \in K$$

And hence $\overline{\mathcal{W}}_{x_0}$ is abelian. Further we have that T_{x_0} gives a unit element, where x_0 denotes the constant hoop at the basepoint, since for all $A \in \mathcal{A}$:

$$(T_{x_0} T_\alpha)[A] = \frac{1}{2}\left(T_\alpha[A] + T_{\alpha^{-1}}[A]\right) = T_\alpha[A]$$

4.) This is not complicated to show, but therefore requires some amount of formal manipulations. Therefore the proof is omitted at this place. □

6.5. Wilson Algebra as Inductive Limit C^*-Algebras

In this section the Wilson algebra will be expressed as an inductive limit of simpler algebras. Therefore first the label set is defined and afterwards the inductive limit decomposition is performed in the second subsection.

6.5.1. The Label Set

We need first some preliminary definitions:

Definition 42 (cp. [6])
Let anything be as before and $x_0 \in \Sigma$ arbitrary but fixed. Then:

1. *A n-tuple of loops $(c_1, ..., c_n) \in \mathcal{C}^n_{x_0}$ is called independent, iff:*
   a) *For each c_i there is an open segment $R_i \subset c_i([0,1])$ s.th. $\forall p \in R_i : \exists_1 t \in [0,1] : c_i(t) = p$.*
   b) *$\forall i, j = 1, ..., n : R_i \cap R_j$ is finite.*
2. *A n-tuple of hoops $(\alpha_1, ..., \alpha_n) \in \mathcal{Q}_{x_0}$ is called independent, iff there exists an independent n-tuple of loops $(c_i)^n_{i=1} \in \mathcal{C}^n_{x_0}$ with $c_i \in \alpha_i$.*
3. *A subgroup $S \subset \mathcal{Q}_{x_0}$ is called tame, if it is generated by a finite number of independent hoops.*
4. *Let $(\alpha_1, ..., \alpha_n)$ be a n-tuple of independent hoops. Then denote the tame subgroup generated by those hoops as:*
$$S = S(\alpha_1, ..., \alpha_n)$$

With this we now define the used label set:

Definition 43 (cp. [6])
We define the tuple $(L, \leq)$ via:

1. $L = \{S \subset \mathcal{Q}_{x_0} | S \text{ is tame subgroup}\}$.
2. *$S_1 \leq S_2 \in L \Leftrightarrow S_1$ is subgroup of S_2.*

That L is indeed a label set, is shown in Thm. 6.2.25 of [31].

Lemma 33 (cp. Thm 6.2.25 of [31])
$(L, \leq)$ is a partially ordered, directed set.

6.5.2. The Inductive Family of C^*-algebras

We now define an inductive family of C^*-algebras.

Definition 44
Let $\overline{\mathcal{W}}_{x_0}(S)$ be the C^*-algebra defined via

$$\mathcal{W}_{x_0}(S) = FS/K_S$$

where FS is the free vector-space over S and K_S is defined as:

$$K_S = \left\{ \sum_{i=1}^{n} z_i T_{\alpha_i} \in FS \middle| \forall A \in \mathcal{A} : \sum_{i=1}^{n} z_i T_\alpha[A] = 0 \right\}$$

And where the norm and the $*$-algebra structure are defined in the same way as for $\overline{\mathcal{W}}_{x_0}$. (Please note, that elements of FS corresponding to $\alpha \in S$ are again denoted by T_α).

We now show, that this gives really an inductive family of C^*-algebras:

Proposition 6
The family $(\overline{\mathcal{W}}_{x_0}(S), \phi_{S'S})_{S,S' \in L}$ defined via

$$\phi_{S,S'} : \mathcal{W}_{x_0}(S) \to \mathcal{W}_{x_0}(S'), T_\alpha \mapsto T_\alpha$$

is an inductive family of C^*-algebras. Further its inductive limit is given by $\overline{\mathcal{W}}_{x_0}$ together with the maps:

$$\phi_S : \mathcal{W}_{x_0}(S) \to \mathcal{W}_{x_0}, T_\alpha \mapsto T_\alpha$$

Proof. That each $\overline{\mathcal{W}}_{x_0}(S)$ defines a C^*-algebra is shown in the same way, as it was shown, that $\overline{\mathcal{W}}_{x_0}$ defines a C^*-algebra. Therefore the corresponding proof is omitted here. We now show, that $\phi_{S'S}$ defines an isometric $*$-morphism. Therefore again lemma 38 is used. We show first, that $K_S \subset \ker(\phi_{S'S})$. Therefore let $x = \sum_{i=1}^m z_i T_{\alpha_i} \in K_S$, i.e. $\forall A \in \mathcal{A} : x[A] = \sum_{i=1}^m z_i T_{\alpha_i}[A] = 0$. Now we have:

$$\phi_{S'S} x = \sum_{i=1}^{m} z_i T_{\alpha_i}$$

And hence

$$\forall A \in \mathcal{A} : (\phi_{S'S} x)[A] = \sum_{i=1}^{m} z_i T_{\alpha_i}[A] = x[A] = 0$$

and hence $K \subset \ker(\phi_{S'S})$. That it defines an isometry on $\mathcal{W}_{x_0}(S)$ follows also easy. Therefore let $\sum_{i=1}^m z_i T_{\alpha_i} \in \mathcal{W}_{x_0}(S)$. Then:

$$\begin{aligned}
\|\phi_{S'S} \sum_{i=1}^{m} z_i T_{\alpha_i}\| &= \sup_{A \in \mathcal{A}} \left| \sum_{i=1}^{m} z_i T_{\alpha_i}[A] \right| \\
&= \| \sum_{i=1}^{m} z_i T_{\alpha_i} \|
\end{aligned}$$

Now observe further:

$$\begin{aligned}\phi_{S'S}(T_\alpha T_\beta) &= \frac{1}{2}\phi_{S'S}(T_{\alpha\circ\beta} + T_{\alpha\circ\beta^{-1}})\\ &= \frac{1}{2}\left(T_{\alpha\circ\beta} + T_{\alpha\circ\beta^{-1}}\right)\\ &= \phi_{S'S}(T_\alpha)\phi_{S'S}(T_\beta)\end{aligned}$$

$$\phi_{S'S}(T_\alpha^*) = \phi_{S'S}(T_\alpha) = T_\alpha = T_\alpha^* = \phi_{S'S}(T_\alpha)^*$$

Hence lemma 38 gives, that $\phi_{S'S}$ defines an unique isometric $*$-morphism. We now show, that $\phi_{S'S}$ satisfies the composition properties demanded in the definition of an inductive family. Therefore observe:

$$\begin{aligned}\phi_{S''S'} \circ \phi_{S'S} &: T_\alpha \mapsto T_\alpha\\ \phi_{SS} &= T_\alpha \mapsto T_\alpha\end{aligned}$$

And observe further:

$$\begin{aligned}\phi_{S''S} &: T_\alpha \mapsto T_\alpha\\ id &: T_\alpha \mapsto T_\alpha\end{aligned}$$

And since by lemma 38 an $*$-morphism is uniquely determined by its action on the basis elements, the claim follows.

We now want to show, that $\lim_\rightarrow \overline{\mathcal{W}}_{x_0}(S) = \overline{\mathcal{W}}_{x_0}$.

Since $\mathcal{W}_{x_0}(S) \subset \mathcal{W}_{x_0}$ holds, we have, that the inclusion is an isometry and hence extends to an isometric inclusion $\overline{\mathcal{W}}_{x_0}(S) \subset \overline{\mathcal{W}}_{x_0}$ using lemma 9. Further it follows by the same argumentation, that $\overline{\mathcal{W}}_{x_0}(S) \subset \overline{\mathcal{W}}_{x_0}(S')$ for $S \leq S'$. Hence we have with lemma 11:

$$\lim_{\rightarrow} \overline{\mathcal{W}}_{x_0}(S) = \overline{\bigcup_{\gamma\in L} \overline{\mathcal{W}}_{x_0}(S)} = \overline{\bigcup_{\gamma\in L} \mathcal{W}_{x_0}(S)}$$

Now the claim would follow, if $\bigcup_{S\in L} \mathcal{W}_{x_0}(S) = \mathcal{W}_{x_0}$ holds. This is indeed the case: $\subseteq$ follows directly since $\bigcup_{S\in L} S \subset \mathcal{Q}_{x_0}$. The other direction follows, since each element $N \in \mathcal{W}_{x_0}$ can be written as $N = \sum_{i=1}^m z_i T_{\alpha_i}$ for some independent hoops$(\alpha_i)_{i=1}^m$ and hence $N \in \mathcal{W}_{x_0}(S)$ with $S = S(\alpha_1, ..., \alpha_m)$. □

6.6. The Space of Generalized Connections and Projective Limits

Since the topological side of the correspondence is analyzed in depth in standard LQG literature, we don't give proofs in this section. The corresponding proofs can all be found in chapter 6.2 of [31].

We define the the following projective family of compact Hausdorff spaces:

Definition 45 (cp. Def. 6.2.14 and Def. 6.2.15 of [31])
Let anything be as before. Then: Define a projective family of compact Hausdorff spaces $(X_S, p_{SS'})_{S,S'\in L}$ *via*

$$\begin{aligned}X_S &:= \mathrm{Hom}(S, SU(2))\\ p_{SS'} &: X_{S'} \to X_S, \Psi \mapsto \Psi|_S\end{aligned}$$

where the topology on X_S is given by the canonical isomorphism $\Psi \in \mathrm{Hom}(S, SU(2)) \mapsto (\Psi(\alpha_1), ..., \Psi(\alpha_n)) \in SU(2)^n$ for $S = S(\alpha_1, ..., \alpha_n)$.

Lemma 34 (cp. p. 172 of[31])
Let anything be as in the last definition. Then:

1. *The map $\Psi \in \mathrm{Hom}(S, SU(2)) \mapsto (\Psi(\alpha_1), ..., \Psi(\alpha_n)) \in SU(2)^n$ for $S = S(\alpha_1, ..., \alpha_n)$ is an isomorphism.*

2. *$(X_S, p_{SS'})_{S,S' \in L}$ is a projective family of compact Hausdorff spaces.*

We have further:

Theorem 41 (cp. p. 178 of [31])
The projective limit of the projective family $(X_S, p_{SS'})_{S,S' \in L}$ is given by

$$\lim_{\leftarrow} X_S = Hom(\mathcal{Q}_{x_0}, SU(2))$$

together with the maps

$$p_S : Hom(\mathcal{Q}_{x_0}, SU(2)) \to X_S, \Psi \mapsto \Psi|_S$$

and is called the space of generalized connections.

We now want to endow X_S with a projective family of group actions.

Proposition 7 (cp. p. 178 of [31])
Define for each $S \in L$ the map

$$\tilde{\Phi}_S : X_S \times SU(2) \to X_S, \Psi \mapsto \Psi.g := g^{-1}\Psi(\cdot)g$$

Then $(\tilde{\Phi}_S)_{S \in L}$ is a projective family of group actions corresponding to the projective family of compact Hausdorff spaces $(X_S, p_{SS'})_{S,S' \in L}$. Its projective limit is given by:

$$\tilde{\Phi} : \mathrm{Hom}(\mathcal{Q}_{x_0}, SU(2)) \times SU(2) \to \mathrm{Hom}(\mathcal{Q}_{x_0}, SU(2)), (\Psi, g) \mapsto \Psi.g := g^{-1}\Psi(\cdot)g$$

6.7. The Wilson Algebras as Subalgebras of $C(X_S)$

The aim of this section is to obtain a canonical embedding $\Gamma : \overline{\mathcal{W}}_{x_0}(S) \hookrightarrow C(X_S)$ and to show, that under this embedding $\overline{\mathcal{W}}_{x_0}(S) \cong \mathrm{Fix}_{SU(2)}(C(X_S))$ holds. Then the structure of the quantum configuration space of loop quantum gravity follows easily using the results of chapter 3 and chapter 4.

We now first need the following Lemma:

Lemma 35 (cp. Lemma 3.3 of [4])
The map $\mathcal{A} \to X_S$ defined by $A \mapsto (H.[A] : \alpha \mapsto H_\alpha[A])$ is surjective.

Proof. See Lem 3.3 of [4]. □

We now define the conjectured image of Γ_S in $C(X_S)$:

Definition 46
Define a subalgebra $\overline{\mathcal{V}}_{x_0}(S) \subset C(X_S)$ *via*

$$\mathcal{V}_{x_0}(S) = \mathrm{lin}\{\tilde{T}_\alpha : X_S \to \mathbb{R}, \Psi \mapsto \mathrm{tr}\,(\Psi(\alpha))\,|\alpha \in S\}$$

together with the $*$*-structure and the norm inherited from* $C(X_S)$.

We now show that this defines really a subalgebra and further, that a $\Gamma : \overline{\mathcal{W}}_{x_0}(S) \hookrightarrow C(X_S)$ with $\mathrm{im}(\Gamma) = \overline{\mathcal{V}}_{x_0}(S)$ exists:

Lemma 36
Let anything be as before. Then:

1. $\overline{\mathcal{V}}_{x_0}(S) \subset C(X_S)$ *is a subalgebra.*

2. *The map* $\Gamma : \mathcal{W}_{x_0}(S) \to \mathcal{V}_{x_0}(S), T_\alpha \mapsto \tilde{T}_\alpha$ *extends to an isometric isomorphism of* C^**-algebras.*

Proof. (1.) We first show, that $\tilde{T}_\alpha : X_S \to [-1,+1]$ is continuous for all $\alpha \in S$. Therefore first observe, that for matrix Lie groups the trace is smooth map, since it is a polynomial in coordinates. Now we want to show, that the map $\alpha\lrcorner\cdot : X_S \to SU(2), \Psi \mapsto \Psi(\alpha)$ is continuous. Therefore let $\alpha_1, .., \alpha_n$ be a generating set of S. Hence we can write any $\alpha \in S$ as a finite product of the generators:

$$\alpha = \prod_{i=1}^{m} \alpha_{j_i}^{(-1)^{n_i}}$$

We now apply the canonical isomorphism $X_S \cong SU(2)^n$ which identifies $\Psi \in X_S$ with a tuple $(g_i)_{i=1}^n \subset SU(2)^n$. With this we have $\Psi(\alpha) = \prod_{i=1}^m g_{j_i}^{(-1)^{n_i}}$. All together:

$$\alpha\lrcorner\cdot : \Psi = (g_i)_{i=1}^n \mapsto \prod_{i=1}^{m} g_{j_i}^{(-1)^{n_i}}$$

But this is continuous, since the multiplication and the inverse are continuous operations in topological groups. Hence we have shown, that $\mathcal{V}_{x_0}(S) \subseteq C(X_S)$ and hence $\overline{\mathcal{V}}_{x_0}(S) \subset C(X_S)$ since $C(X_S)$ is complete. That the sup norm satisfies the C^*-property on $\mathcal{V}_{x_0}(S)$ can be shown in the same way as the analogous statement was shown in proposition 6. Hence its completion is a C^*-algebra.
2.) We first show, that $\ker(\Gamma) = K_S$. Therefore let $x = \sum_i z_i T_{\alpha_i} \in \mathcal{W}_{x_0}(S)$. Now let $\Gamma(\sum_i z_i T_{\alpha_i}) = 0$, i.e. $\forall \Psi \in X_S : \sum_i z_i \tilde{T}_{\alpha_i}[\Psi] = 0$. By lemma 35 we then have, that this is equivalent to:

$$\forall A \in \mathcal{A} : \sum_i z_i \mathrm{tr}\,(h_{\alpha_i}[A]) = 0 \Leftrightarrow \forall A \in \mathcal{A} : \sum_i z_i T_{\alpha_i}[A] = 0$$
$$\Leftrightarrow \sum_i z_i T_{\alpha_i} \in K$$

We now show, that Γ is an isometry:

$$\begin{aligned}
\left\| \Gamma\left(\sum_i z_i T_{\alpha_i}\right) \right\| &= \sup_{\Psi \in X_S} \left| \sum_i z_i \tilde{T}_{\alpha_i}[\Psi] \right| \\
&= \sup_{A \in \mathcal{A}} \left| \sum_i z_i \mathrm{tr}(H_{\alpha_i}[A]) \right| \\
&= \sup_{A \in \mathcal{A}} \left| \sum_i z_i T_{\alpha_i}[A] \right| \\
&= \left\| \sum_i z_i T_{\alpha_i} \right\|
\end{aligned}$$

Further we have

$$\begin{aligned}
\Gamma(T_\alpha T_\beta) &= \Gamma\left(\frac{1}{2}(T_{\alpha\circ\beta} + T_{\alpha\circ\beta^{-1}})\right) \\
&= \frac{1}{2}\Gamma(T_{\alpha\circ\beta}) + \frac{1}{2}\Gamma(T_{\alpha\circ\beta^{-1}}) \\
&= \frac{1}{2}\left(\mathrm{tr}(\cdot(\alpha\circ\beta)) + \mathrm{tr}(\cdot(\alpha\circ\beta^{-1}))\right) \\
&= \mathrm{tr}(\cdot(\alpha))\mathrm{tr}(\cdot(\beta)) \\
&= \Gamma(T_\alpha)\Gamma(T_\beta)
\end{aligned}$$

where we have used the Mandelstam identities. And also

$$\Gamma(T_\alpha^*) = \Gamma(T_\alpha) = \Gamma(T_\alpha)^*$$

by proposition 3. Hence it defines an isometric $*$-morphism by lemma 38. That it is surjective follows directly by the construction, since $\tilde{T}_\alpha \mapsto T_\alpha$ defines an inverse. Hence we have shown, that it Γ defines an isometric, bijective $*$-morphism, i.e. an isometric $*$-isomorphism. □

We now finally show:

Theorem 42
Let anything be as before. Then:

$$\overline{\mathcal{V}}_{x_0}(S) = \mathrm{Fix}_{SU(2)}(C(X_S))$$

Proof. We use theorem 19 to prove this theorem.

Assertion 1: $\overline{\mathcal{V}}_{x_0}(S) \subseteq \mathrm{Fix}_{SU(2)}(C(X_S))$.

Proof of Assertion 1: We have, that any element of $\mathcal{V}_{x_0}(S)$ is a finite linear combination of $\tilde{T}_\alpha$. Therefore it suffices to show, that $\tilde{T}_\alpha \in \mathrm{Fix}_{SU(2)}(C(X_S))$ holds for any $\alpha \in S$. But this follows most easily by the cyclicity of the trace:

$$\forall \Psi \in X_S : \tilde{T}_\alpha(\Psi.g) = \tilde{T}_\alpha(g^{-1}\Psi(\cdot)g) = \tilde{T}_\alpha(gg^{-1}\Psi(\cdot)) = \tilde{T}_\alpha(\Psi)$$

Then the assertion follows, since $\mathrm{Fix}_{SU(2)}(C(X_S))$ is a closed subalgebra.

Assertion 2: $\pi_* : \overline{\mathcal{V}}_{x_0}(S) \to C(X_S/G), N \mapsto N \circ \pi^{-1}$ is a well-defined, isometric $*$-morphism.

Proof of Assertion 2: We first show, that it is well-defined, i.e. that $\operatorname{im}(\pi_*) \subset C(X_S/G)$. Therefore let $[\Psi] \in X_S/G$ and $\Psi, \tilde{\Psi} \in \pi^{-1}([\Psi])$, i.e. $\exists g \in SU(2)$ with $\Psi = \tilde{\Psi}.g$. Now let $N \in \overline{\mathcal{V}}_{x_0}(S)$. Assertion 1 then gives $N \in \operatorname{Fix}_{SU(2)}(C(X_S))$ and hence we have:

$$N(\Psi) = N(\tilde{\Psi}.g) = N(\tilde{\Psi})$$

Further, $\pi : X_S \to X_S/G$ is open in the case of group actions by compact Hausdorff groups (cp. Lem. 3.7.11 of [16]) and hence π^{-1} is continuous. Since N is continuous it then follows, that $N \circ \pi^{-1} \in C(X_S/G)$. We now show, that π_* is an isometry. This follows directly by surjectivity of π:

$$\begin{aligned} \|\pi_* N\| &= \sup_{x \in X_S/G} \left| N \circ \pi^{-1}(x) \right| \\ &= \sup_{x \in \pi(X_S)} \left| N \circ \pi^{-1}(x) \right| \\ &= \sup_{x \in X_S} |N(x)| \\ &= \|N\| \end{aligned}$$

And hence it is also injective. That π_* is a $*$-morphism follows also directly, since for $\alpha, \beta \in \mathbb{C}$ and $N_1, N_2 \in \overline{\mathcal{V}}_{x_0}(S) \subset C(X_S)$ we have:

$$\begin{aligned} (N_1 N_2) \circ \pi^{-1} &= \left(N_1 \circ \pi^{-1}\right)\left(N_2 \circ \pi^{-1}\right) \\ (\alpha N_1 + \beta N_2) \circ \pi^{-1} &= \alpha \left(N_1 \circ \pi^{-1}\right) + \beta \left(N_2 \circ \pi^{-1}\right) \\ N_1^* \circ \pi_{-1} &= \left(N_1 \circ \pi^{-1}\right)^* \end{aligned}$$

And hence the assertion is shown.

Assertion 3: $\tilde{\Phi}^U : X_s \curvearrowleft U(2), (\Psi, g) \mapsto \Psi.g := g^{-1}\Psi(\cdot)g$ is a group action such that the restriction on $X_S \times SU(2)$ is given by $\tilde{\Phi}$.

Proof of Assertion 3: Since in the proof of proposition 7 no concrete reference to the group was made and $SU(2) \subset U(2)$, it follows directly, that $\tilde{\Phi}^U$ is a group action. By the concrete form of $\tilde{\Phi}^U$ one sees also directly, that $\tilde{\Phi}^U(\cdot, g) = \tilde{\Phi}(\cdot, g)$ holds for $g \in SU(2) \subset U(2)$, and hence the assertion follows.

Assertion 4: Let $\Psi_1, \Psi_2 \in X_S$ s.th.:

$$\forall \alpha \in S : \tilde{T}_\alpha(\Psi_1) = \tilde{T}_\alpha(\Psi_2) \tag{6.1}$$

Then there is a $g \in U(2)$ s.th. $\Psi_1 = \Psi_2.g$.

Proof of Assertion 4: This was shown in [1]. But for the sake of importance of this theorem, we present the proof here. It is a well-known fact, that for a topological group H there exists a compact Hausdorff group $K(H)$ (called the associated compact group), together with a homomorphism $\kappa : H \to K(H)$ with dense image, which satisfies the following universal property: For any compact Hausdorff group G and any homomorphism $\Lambda : H \to G$ there exists a unique homomorphism $\tilde{\Lambda} : K(H) \to G$ such that $\Lambda = \tilde{\Lambda} \circ \kappa$.

Further for representations of compact groups the following holds (cp. Thm 4.12 of [11]): Let K be a compact group and let V be a finite dimensional vector space. Let $\rho_i : K \to \mathrm{Hom}(V)$ with $i = 1, 2$ be representations. Let further $\forall k \in K : tr(\rho_1(k)) = tr(\rho_2(k))$. Then there is a unitary $U \in \mathrm{Hom}(V, V)$ such that $\rho_1 = U^\dagger \rho_2 U$.

Now let $\Psi_1, \Psi_2 \in X_S = \mathrm{Hom}(S, SU(2))$. Then there are by above $\tilde{\Psi}_1, \tilde{\Psi}_2 \in \mathrm{Hom}(K(S), SU(2))$ such that $\Psi_i = \tilde{\Psi}_i \circ \kappa$ for $i = 1, 2$. With this eq. (6.1) reads:

$$\forall \alpha \in S : \tilde{T}_\alpha(\Psi_1) = \mathrm{tr}(\tilde{\Psi}_1 \circ \kappa(\alpha)) = \mathrm{tr}(\tilde{\Psi}_2 \circ \kappa(\alpha)) = \tilde{T}_\alpha(\Psi_2)$$

Since $\mathrm{im}(\kappa)$ is dense in $K(S)$ and tr is continuous, this implies:

$$\forall \tilde{\alpha} \in K(S) : \mathrm{tr}(\tilde{\Psi_1}(\alpha)) = \mathrm{tr}(\tilde{\Psi_2}(\alpha))$$

Hence by above mentioned theorem, there is a $g \in U(2)$ such that $\Psi_1 = \Psi_2.g$.

Assertion 5: Define the group actions $Ad_U : SU(2) \curvearrowleft U(2)$ and $Ad_{SU} : SU(2) \curvearrowleft SU(2)$ by

$$\begin{aligned} Ad_U &: SU(2) \times U(2) \to SU(2), (h, g) \mapsto g^{-1}hg \\ Ad_{SU(2)} &: SU(2) \times SU(2) \to SU(2), (h, g) \mapsto g^{-1}hg \end{aligned}$$

Then: $SU(2)/U(2) \cong SU(2)/SU(2)$.

Proof of Assertion 5: This is also proved in [1]. But for the sake of importance of this theorem, we present the proof here, too. Therefore define first a smooth group homomorphism:

$$A : U(1) \times SU(2) \to U(2), (z, S) \mapsto zS$$

We now show, that this homomorphism is surjective: Let $h \in U(2)$. Now set $\delta = \sqrt{\det(h)}$. Observe that, since $h \in U(2)$ we have, that $\delta \in U(1)$. Then: $A(\delta, \delta^{-1}h) = h$ and further $\det(\delta^{-1}h) = \delta^{-n}\det(h) = +1$ and hence $\delta^{-1}h \in SU(2)$. We now want to show:

$$\forall g_1, g_2 \in SU(2) \exists h \in SU(2) : g_1 = h^{-1}g_2h \Leftrightarrow \exists h \in U(2) : g_1 = h^{-1}g_2h$$

"$\Rightarrow$" is clear, since $SU(2) \subset U(2)$. Now towards "$\Leftarrow$". Therefore let $h \in U(2)$ with $g_1 = h^{-1}g_2h$. Then, since A is onto, we have $(z, s) \in U(1) \times SU(2)$ with $h = zS$. Now $h^{-1} = S^{-1}\bar{z}$ and further, since z and $\bar{z}$ commute with matrices, we have $g_1 = S^{-1}g_2Sz\bar{z} = S^{-1}g_2S$ and hence there is a $h \in SU(2)$ with $g_1 = h^{-1}g_2h$. Hence the assertion follows with lemma 23.

Proof of the Theorem: By Assertion 5 we have an homeomorphism $SU(2)/SU(2) \cong SU(2)/U(2)$. This gives an homeomorphism $X_S/SU(2) \cong X_S/U(2)$ using the canonical homeomorphism $X_S \cong SU(2)^{n_S}$ where n_S is the number of independent generators of S. Now $(\tilde{T}_\alpha)_{\alpha \in S}$ seperates points on $X_S/U(2)$ by Assertion 4 and hence it does so also on $X_S/SU(2)$. Finally it remains to show, that there is an $\alpha \in S$ such that $\mathrm{tr}(\Psi(\alpha)) \neq 0$ holds for all $\Psi \in X_S$. Therefore set $\alpha = e_S$, the identity element of S. Since group homomorphisms preserve identities, we have $x(\alpha) = e_{SU(2)}$ and hence $\mathrm{tr}(x(\alpha)) = 1 \neq 0$. Hence we can apply theorem 19 which gives, that $\overline{\mathcal{V}}_{x_0}(S) = C(X_S/SU(2)) = \mathrm{Fix}_{SU(2)}(C(X_S))$, and hence the claim follows. $\square$

Now we have the following easy Corollar:

Corollar 8
Let anything be as before. Then:

$$\Delta(\overline{\mathcal{W}}_{x_0}) = \mathrm{Hom}(\mathcal{Q}_{x_0}, SU(2))/SU(2)$$

Proof. With theorem 42 and lemma 36 we have, that

$$\Delta(\overline{\mathcal{W}}_{x_0}(S)) = \Delta(\mathrm{Fix}_{SU(2)}(C(X_S)))$$

and hence with theorem 28 we have:

$$\Delta(\overline{\mathcal{W}}_{x_0}(S)) = X_S/G$$

Using theorem 33 and theorem 30 we obtain unambiguosly:

$$\Delta(\overline{\mathcal{W}}_{x_0}) = \lim_{\leftarrow}(X_S/G) = \left(\lim_{\leftarrow} X_S\right)/G$$

And hence by theorem 41 and proposition 7 we have finally:

$$\Delta(\overline{\mathcal{W}}_{x_0}) = \mathrm{Hom}(\mathcal{Q}_{x_0}, SU(2))/SU(2)$$

Where the quotient is performed with respect to the inductive limit group action $\Psi \mapsto \Psi.g = g^{-1}\Psi(\cdot)g$. □

6.8. Discussion

In this section the relevance of the concepts developed in part I were illustrated by applying them to the calculation of the quantum configuration space of loop quantum gravity. The quantum configuration space obtained with our methods equals the result presented in the literature (see e.g. [1], [31]). Further the concepts of part I give some further result: On the one hand it follows from theorem 27 that each cyclic representation of the point holonomy algebra arises as an inductive limit of representations of the corresponding inductive families. Further it follows with theorem 14, that an orthonormal basis for the inductive limit representation is determined by the orthonormal bases of the corresponding members of the inductive family.

7. Summary and Outlook

In the first part of the thesis a comprehensive and coherent treatment of the spectral theory of inductive limit C^*-algebras and C^*-dynamical systems was presented. Here the most important results regarding inductive limit C^*-algebras are:

- A projective limit of compact Hausdorff spaces corresponds always dually to an inductive limit of unital, abelian C^*-algebras:
  - Given an inductive family of C^*-algebras, their spectra form a projective family. Further its projective limit is given by the spectrum of the inductive limit C^*-algebra (cp. theorem 18). This result can be also found in [30].
  - Given a projective family of compact Hausdorff spaces, the algebras of continuous functions thereon form an inductive family of C^*-algebras. Further its inductive limit is given by the algebra of continuous functions on the projective limit of the topological spaces (cp. theorem 20).
  - The Gel'fand transform is compatible with inductive limits of C^*-algebras and inductive limits of algebras of continuous functions on their spectra and makes the above explained correspondence explicit (cp. lemma 17 and theorem 25).
- Each cyclic representation of an inductive limit of unital, abelian C^*-algebras arises as an inductive limit of certain cyclic representations of the inductive family (cp. theorem 26).

The most important results regarding C^*-dynamical systems are:

- A C^*-dynamical system corresponds always dually to a compact Hausdorff space together with a continuous group action:
  - Given a C^*-dynamical system, the C^*-group action induces a continuous group action on the spectrum of the C^*-algebra (cp. theorem 28).
  - Given a compact Hausdorff space together with a continuous group action, the algebra of continuous functions thereon becomes a C^*-dynamical system (cp. lemma 21).
- The spectrum of the fixed point algebra of a C^*-dynamical system (i.e. the algebra of invariant elements) is given by the quotient of the spectrum of the full algebra by the induced group action on the spectrum (cp. theorem 28).

The most important results corresponding the compatibility of C^*-dynamical systems with inductive limits are:

- Inductive families and inductive limits can be consistently defined for C^*-dynamical systems (cp. definition 27 and theorem 31).
- The inductive limit of the fixed point subalgebras of an inductive family of C^*-dynamical systems is given by the fixed point subalgebra of the inductive limit C^*-dynamical system (cp. theorem 33).

In the second part of the thesis the results of the first part were applied to polymer quantization of the real scalar field and loop quantization of gravity. The main result was in both cases, that an inductive limit decomposition of the corresponding quantum algebras can be performed (cp. section 5.3 and section 6.5). Furthermore the theory of C^*-dynamical systems was useful in the case of loop quantum gravity (cp. theorem 42 and corollar 8). Further the methods used in this thesis yield the correct quantum configuration spaces as discussed in section 5.6 and section 6.8.

In the following we want to present possible directions for further research on these topics:

- Concerning the theory of inductive limits of C^*-algebras, in this thesis it has been always assumed, that the morphisms are surjective on the topological side and isometric on the C^*-side. It would be interesting to analyze, to which extent the results in this thesis would hold, if one considers more general maps. Further in the second chapter it was mentioned, that it might be possible, that a cyclic representation arises as an inductive limit of non-cyclic representations (cp. discussion under the proof of theorem 26). A further analysis of this case aiming at existence and possible physical applications (maybe in the context of the further down mentioned case of algebras of local observables) might be interesting.

- Concerning the theory of C^*-dynamical systems, the question arose, in which situations it is possible to embed a given C^*-algebra as a fixed point subalgebra into a C^*-dynamical system and further, which freedom of choice for the C^*-dynamical system one has in this situation. Maybe this also has an application to gauge theory, since embeddings of a given algebra of gauge invariant observables in different C^*-dynamical systems would correspond to a duality between different gauge theories.

- Concerning the theory of inductive limits of C^*-dynamical systems, we assumed in this thesis, that all members of the inductive family share the same transformation group. It would be interesting to relax this assumption by using projective family of groups as the corresponding transformation groups. This program has been investigated in special cases on the topological side (cp. Ch. 2 of [31], [5]), but it might be useful, to embed it in the comprehensive framework presented in this thesis.

- Finally, loop quantum gravity is not the only place in physics, were inductive and projective limits appear. For example inductive limits appear also in algebraic quantum field theory, where the algebra of all local observables can be constructed using an inductive limit of the inductive family of algebras of observables bounded to finite regions (cp. [14]). In this case the label set corresponds to all finite regions of spacetime. Further, there have been attempts to formulate renormalization theory or coarse graining procedures in terms of inductive limits (cp. [29]). In both cases it would be interesting to analyze, if the spectral theory of inductive limit C^*-algebras as presented in this thesis would lead to any valuable perspective.

A. On "Freely Generated" $*$-Algebras

The C^*-algebras occuring in Chapter 5 and Chapter 6 where defined by using free vector spaces and relations. The corresponding theory is excluded in this appendix.

We first define the notion of a free vector space over a set:

Definition 47 (cp. [23])
Let S be a set. Then define the complex free vector space over S written as FS as follows: Set

$$FS = \{\phi : S \to \mathbb{C} | \phi(s) = 0 \text{ for almost all } s \in S\}$$

Further define for $z \in \mathbb{C}$, $x \in S$ the map $zx \in FS$ via:

$$zx : S \to \mathbb{C}, s \mapsto \begin{cases} z & \text{if } s = x \\ 0 & \text{if } s \neq x \end{cases}$$

We define an scalar multiplication on FS via $\forall \alpha \in \mathbb{C} : \forall \phi \in FS : (\alpha\phi : s \mapsto \alpha\phi(s))$ and an vector-addition via $\forall \phi, \psi \in FS : (\phi + \psi : s \mapsto \phi(s) + \psi(s))$. Further it can be shown, that FS is a vector-space and that the family of maps $\forall x \in S : \delta_x := 1x$ is a basis for FS. Hence any element $v \in FS$ can be written uniquely as a finite sum

$$v = \sum_{i=1}^{m} z_i s_i$$

with $s_i \in S$ and $z_i \in \mathbb{C}$.

We now want to understand, how a free vector space can be upgraded to a operator algebra:

Lemma 37
Let S be a set and FS be the free complex vector space over S. Then:

1. *Let $\cdot : S \times S \to S$ be an associative map, i.e. $\forall s_1, s_2, s_3 \in S : (s_1 \cdot s_2) \cdot s_3 = s_1 \cdot (s_2 \cdot s_3)$. Then FS together with the product*

$$\cdot : FS \times FS \to FS, \left(\sum_{i=1}^{n} x_i s_i\right) \left(\sum_{j=1}^{m} y_j s_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j (s_i \cdot s_j)$$

   defines an associative algebra.

2. *Let $\cdot : S \times S \to S$ be an associative map as in (1). Let $* : S \to S$ be an involutive map, i.e. $\forall s_1, s_2 \in S : (s_1 \cdot s_2)^* = s_2^* \cdot s_1^*$ and $\forall s \in S : (s^*)^* = s$. Then $(FS, \cdot)$ as defined in (1) together with*

$$* : FS \to FS, \left(\sum_{i=1}^{n} z_i s_i\right)^* = \sum_{i=1}^{n} \bar{z}_i s_i^*$$

   defines an involution on $(FS, \cdot)$ and makes $(FS, \cdot, *)$ to a $*$-algebra.

3. *Let X be an $*$-algebra and $J \subset X$ be an non-trivial 2-sided ideal with $J^* = J$. Then X/J is a $*$-algebra.*

Proof. 1.) We first show associativity. In the following all sums should be understood as finite sums.

$$\begin{aligned}
\left[\left(\sum_i z_i e_i\right)\left(\sum_j z_j e_j\right)\right]\left(\sum_k z_k e_k\right) &= \left[\sum_{ij} z_i z_j (e_i \cdot e_j)\right]\sum_k z_k e_k \\
&= \sum_{ijk} (z_i z_j) z_k (e_i \cdot e_j) \cdot e_k \\
&= \sum_{ijk} z_i (z_j z_k) e_i \cdot (e_j \cdot e_k) \\
&= \left(\sum_i z_i e_i\right)\left[\left(\sum_j z_j e_j\right)\left(\sum_k z_k e_k\right)\right]
\end{aligned}$$

and hence it is associative. Distributivity follows also directly:

$$\left(\alpha \sum_i z_i e_i + \beta \sum_j z_j e_j\right)\left(\sum_k z_k e_k\right) = \alpha\left(\sum_i z_i e_i\right)\left(\sum_k z_k e_k\right) + \beta\left(\sum_j z_j e_j\right)\left(\sum_k z_k e_k\right)$$

and the proof goes for multiplication from the other side analogously.
2.) We have:

$$\left(\sum_i z_i e_i\right)^{**} = \left(\sum_i \bar{z}_i e_i^*\right)^* = \sum_i z_i e_i$$

and

$$\begin{aligned}
\left[\left(\sum_i z_i e_i\right)\left(\sum_j z_j e_j\right)\right]^* &= \sum_{ij} \bar{z}_i \bar{z}_j (e_i \cdot e_j)^* \\
&= \sum_{ij} \bar{z}_j \bar{z}_i \left(e_j^* \cdot e_i^*\right) \\
&= \left(\sum_j z_j e_j\right)^*\left(\sum_i z_i e_i\right)^*
\end{aligned}$$

and finally antilinearity follows directly by definition and hence the assertion follows.
3.) See [26]. □

Lemma 38
Let S_1, S_2 be two sets, FS_1, FS_2 be the corresponding free complex vector spaces. Let further FS_1 and FS_2 be endowed with a $*$-algebra structure. Then: Let $\phi : S_1 \to FS_2$ be a map such that $\forall s_1, s_2 \in S_1 : \phi(s_1 \cdot s_2) = \phi(s_1) \cdot \phi(s_2)$ and $\phi(s_1^*) = \phi(s_1)^*$. Then ϕ extends to a unique $*$-morphism $\phi : FS_1 \to FS_2$ via

$$\phi : \sum_i z_i s_i \mapsto \sum_i z_i \phi(s_i)$$

Proof. ϕ is linear by construction. Further we have:

$$\phi\left(\left(\sum_i z_i e_i\right)^*\right) = \sum_i \bar{z}_i \phi(e_i^*) = \sum_i \bar{z}_i \phi(e_i)^*$$

and

$$\begin{aligned}
\phi\left(\left(\sum_i z_i e_i\right)\left(\sum_j z_j e_j\right)\right) &= \phi\left(\sum_{ij} z_i z_j (e_i \cdot e_j)\right) \\
&= \sum_{ij} z_i z_j \phi(e_i \cdot e_j) \\
&= \sum_{ij} z_i z_j \phi(e_i)\phi(e_j) \\
&= \left(\sum_i z_j \phi(e_i)\right)\left(\sum_j z_j \phi(e_j)\right)
\end{aligned}$$

To show uniqueness let $\tilde{\phi} : S_1 \to FS_2$ be a linear map with $s_i \in S_1 \mapsto \phi(s_i) \in FS_2$. But then by linearity

$$\tilde{\phi}\left(\sum_i z_i s_i\right) = \sum_i z_i \tilde{\phi}(s_i) = \phi\left(\sum_i z_i s_i\right)$$

and hence ϕ is unique. □

Lemma 39
Let X_1, X_2 be $*$-algebras and $J_1 \subset X_1$, $J_2 \subset X_2$ self-adjoint (i.e. $J_i^* = J_i$), 2-sided ideals. Then a $*$-morphism $\phi : X_1 \to X_2$ induces a $*$-morphism $\tilde{\phi} : X_1/J_1 \to X_2/J_2, [x] \mapsto [\phi(x)]$ if $\phi(J_1) \subset J_2$.

Proof. We first show, that $\tilde{\phi}$ is well defined. Therefore observe, that left-totality follows since the canonical projection $\pi_1 : X_1 \to X_1/J_1$ is surjective. Now we show, that it $\tilde{\phi}$ is right unique: Therefore let $x_1, x_2 \in [x]$. This means, that there is a $j \in J_1$ with $x_1 = x_2 + j$. Then $\phi(x_1) = \phi(x_2 + j) = \phi(x_2) + \phi(j)$ by linearity of ϕ. Now $\phi(j) \in J_2$ by prerequisit, and hence $\pi_2(\phi(x_1)) = \pi_2(\phi(x_2))$ for the canonical projection $\pi_2 : X_2 \to X_2/J_2$. Hence it is well defined. Linearity follows directly, since π_1, π_2 and ϕ are linear by definition. That $\tilde{\phi}$ is a $*$-morphism follows directly, since J_i are ideals and self-adjoint. □

We now want to define the concept of a pre C^*-algebra:

Definition 48
A pre C^*-algebra is a $*$-algebra X together with a norm $\|\cdot\|$ on X, which satisfies

1. *$\forall x, y \in X : \|xy\| \leq \|x\|\|y\|$ (submultiplicativity).*
2. *$\forall x, y \in X : \|x^* x\| = \|x\|^2$ (C^*-property).*

We then have the following final Lemma:

Lemma 40
Let $(X, \|\cdot\|)$ be a pre C^*-algebra. Than $\overline{X}$ is a C^* algebra. Further it is abelian, if X is abelian and unital if X is unital.

Proof. Since the norm is continuous, it follows that the closure of a pre C^*-algebra defines a C^*-algebra. That X is abelian and unital, if it is abelian and unital on a dense subset, follows in the same way, as the corresponding statement was shown in Theorem 16. $\square$

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