On Diffeomorphism Invariance and Black Hole Quantization

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A thesis submitted for the degree of

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We consider the question of the quantization of the black hole area. It is suggested that the physical Hilbert space of quantum microstates of the black hole horizon area can be related with the state space built by the generators of two dimensional diffeomorphism transformations. The spatial constraints of general relativity are, therefore, expanded on a toroidal spacelike surface. The properties of the resulting algebra are explored, and the highest-weight representation space for this algebra is constructed. We argue that the operators of the two dimensional diffeomorphism algebra should be included in the set of operators which are needed for an algebraic description of a quantum black hole. The degeneracy of the black hole horizon area might then be associated to the degeneracy of the operator which is diagonal in the highest-weight representation space. A formal expression for the degeneracy is derived, and its asymptotics might give the correct degeneracy to reproduce the Bekenstein-Hawking entropy formula for black holes.

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Contents

1

Introduction

As challenging as it may be, the problem of the unification of general relativity and quantum mechanics will only be truly solved in combination with some observational confirmation. A theory without experimentally verified predictions will always remain "just a theory". The question of how we are going to know whether a theory of quantum gravity is correct is still an open one. The Planck scale at which the effects of quantum gravity should become relevant is far beyond the reach of current experimental devices. Even the LHC, which is going to reach 10TeV scale, is still 10^{15} orders of magnitude below the Planck scale. Thus, even if one has a mathematically consistent theory of quantum gravity there is no obvious way to test it. This gives rise to a natural question: are there any quantum gravity effects observable at the energies accessible today?

Quantum gravity effects can be very important for black holes. Thus, these are natural candidates to look for possible hints or consistency checks. The question we wish to address in this work is the quantization of the area of black holes. We will begin by briefly recalling what quantum effects are relevant for the black hole physics and what macroscopically observable consequences a discrete area spectrum could have.

1.1 Quantum Effects in Black Holes

The quantum effects, which are relevant for black hole physics are vacuum polarization and particle creation in the presence of an external field [1]. If the external field is strong and can be described classically it is called a *classical background*.

This background changes the vacuum fluctuations of quantum fields by shifting their zero-point energy levels. Hence, the vacuum is "deformed". The shift of the energy levels can be measured and this is called the *vacuum polarization effect*. If, on the other hand, the amount of energy which the quantum field receives from the background field is larger than the difference between the energy levels of the oscillator modes of the quantum field, then we have *particle production* in an external field.

A prominent example of particle creation is the Schwinger effect in quantum electrodynamics, where a positron-electron pair is produced in a strong static electric field. The fact that the "virtual particles" have opposite charge is crucial, as in the external electric field they are moving in opposite directions. In such a way they are separated and can gain a sufficient amount of energy to become real.

In black hole physics both particle production and vacuum polarization effects are important. The vacuum polarization corresponds to the appearance of local terms which modify the gravitational action. In the case of external gravitational field this modification becomes relevant if the curvature of the spacetime approaches the Planck curvature $R_{Pl} = \frac{c^3}{\hbar G} \sim 10^{65}\text{cm}^{-2}$. However, for black holes of mass $M \gg M_{Pl}$ the curvature reaches the Planck scale only "deep inside" the event horizon. Hence, the vacuum polarization effects seem to be negligible outside the black hole.

Let us now consider the possibility of particle creation by a graviational field. The fact that the total energy of the created particle pair has to be zero in gravitational field would imply that one of the particles has negative energy. While negative-energy states can exist in a nonstatic gravitational field, it seems to be impossible to convert a virtual particle-antiparticle pair into a pair of real particles in a static gravitational field. However, Hawking predicted [2] that nonrotating black holes emit radiation with a black body thermal spectrum of temperature

$$T_H = \frac{\kappa}{2\pi} \tag{1.1}$$

and thus evaporate. This implies a probability $w \sim \exp(-E/kT_H)$ of finding an emitted particle with energy E. This probability corresponds to that of particle pair production as a result of vacuum quantum fluctuations in a gravitational field of strength κ. The quantity κ is called the black hole surface gravity and it is equal

to $1/(4M)$ for nonrotating black holes. Hawking radiation is therefore an example of a purely quantum effect which should be detectable for an observer outside the black hole.

In order for a black hole to have significant Hawking radiation within the lifetime of our universe its initial mass has to be smaller than $\sim 10^{15}$g (compare with the solar mass $M_\odot = 2 \cdot 10^{33}$g). Such black holes are called *primordial black holes* as they could have been formed only in the early universe. In the present-day universe these black holes could radiate with sufficiently high temperature to be detected. However, there is currently no evidence for the existence of primordial black holes, and therefore no observational verification of the Hawking effect has been found.

Nevertheless, it seems that Hawking radiation is indeed one of the most important predictions for quantum effects in gravity which could in principle be observable today. Still, the real nature of Hawking radiation at the quantum level is not yet unambigously established. This is because, in order to derive the continuous thermal spectrum for black hole radiation, we are considering quantum fluctuations of matter fields on a classical black hole background. However, in a theory of quantum gravity, quantum fluctuations of the black hole horizon should be taken into account. This indicates that the character of the Hawking radiance spectrum could be modified even for large black holes. In order to investigate these possible modifications of the Hawking radiation due to the effects of quantum gravity, we turn now to the thermodynamics of black holes.

1.2 Thermodynamics of a Quantum Black Hole

Even prior to the discovery of black hole radiation Bekenstein postulated that a black hole possess a certain entropy. This conclusion originated from the "no hair conjecture" [3] which states that a stationary black hole is described only by few parameters: its mass M, angular momentum J, charge Q, and area $A = A(M, J, Q)$. Therefore, if a black hole absorbs matter with certain entropy, then from the point of view of an outside observer the total entropy of the universe would decrease. This would in turn violate the second law of thermodynamics unless the black hole would itself have entropy. Hawking's theorem [4] that the area of a classical black hole is non-decreasing lead Bekenstein [5] to conclude that the black hole entropy should be proportional to its surface area. The proportionality

coefficient $S = A/4$ was fixed only after the prediction of Hawking radiation [6]. This was done by using the following expression, which relates the characteristic parameters of a non-extremal black hole [7]:

$$M^2 = \frac{A}{16\pi}\left(1 + \frac{4\pi Q^2}{A}\right)^2 + \frac{4\pi J^2}{A}. \tag{1.2}$$

Differentiating this relation leads to the analogue of the first law of thermodynamics for black holes

$$dM = \frac{\kappa}{16\pi} dA + \Omega dJ + \phi dQ, \tag{1.3}$$

Here Ω is the angular velocity and ϕ is the electric potential of the black hole. In the prefactor of dA one can recognize the expression for the Hawking temperature (1.1) and read off the proportionality coefficient.

Returning to the quantum description of a black hole, we know that in quantum mechanics both angular momentum and charge can only take discrete values. In combination with equation (1.3) this could be considered as the first indication that the mass and the area of black hole should take discrete values as well. Moreover, the area eigenvalues should be uniformly spaced as $a_n = \alpha l_{Pl}^2 n$ where α is some universal constant [6].

Another justification for a discrete horizon area spectrum was proposed by Mukhanov [8]. He assumed that a black hole is quantized and that every black hole with mass M can be associated with some macrostate at energy level n. In analogy with statistical mechanics one can define the entropy of a particular black hole macrostate as the logarithm of the number of its possible internal configurations $g(n)$:

$$S = \ln g(n). \tag{1.4}$$

The degeneracy $g(n)$ can be identified with number of different ways to reach the level n, starting from the ground state $n = 0$ and then going up the staircase of energy levels in all possible ways. This gives

$$g(n) = 2^{n-1}.$$

For equidistant area levels this leads to the Bekenstein-Hawking entropy formula, and thus justifies the initial assumption that the area spectrum is discrete.

1.3 Observational Consequences of Discrete Area Spectrum

For a nonrotating black hole with zero electric charge, its mass is related to the area as $M^2 = A/(16\pi)$. Hence, a discrete area spectrum implies a discrete mass spectrum $M \sim \sqrt{n}$. It follows that the spectrum of Hawking radiation is not continuous but is instead a line spectrum [8; 9]. Moreover, the energy spacing between consecutive levels corresponds to the frequency $\omega_0 = (8\pi M)^{-1} \ln 2$ for $M \gg M_{Pl}$. The full emission spectrum is then given by spectral lines at frequencies, which are multiples of ω_0, whose envelope is the Hawking thermal spectrum. For primordial black holes this gives a sharp, observable line spectrum as a direct consequence of a discrete and uniform black hole horizon area spectrum.

There is, however, no general agreement on the spacing of the area levels. Several authors (see [10] and references therein) have suggested a non-uniform level spacing. In particular, using the loop quantum gravity approach to black hole physics, Rovelli and Smolin [11; 12] initially proposed the area spectrum

$$A \sim \sum_i \sqrt{j_i(j_i+1)}. \tag{1.5}$$

The index j_i takes integer and half-integer values and labels a *spin-j_i link*, which refers to a possible surface separating two adjacent volume quanta labeled by i. In distinction from the Bekenstein-Mukhanov black hole emission spectrum, the quantum loop area spectrum implies that the spacing between spectral lines is infinitesimal and effectively reproduces the Hawking's thermal spectrum. However, this result and the degeneracy of an area eigenvalue depends very much on the convention about which spin-j_i links are considered to be physically distinguishable and thus have to be taken into account in the sum (1.5). Fully *indistinguishable* links (see ref.[13] for precise meaning of this) fail to reproduce the area-entropy relation, i.e. one gets $S \sim A^t$ with $t < 1$. Moreover, the minimal change in the area is no longer restricted to the Planck area.

After introducing the notion of fully *distinguishable* links, they were able to reproduce the Bekenstein-Hawking entropy law with an equidistant area spectrum $A_j = j + 1/2$ [13]. This agrees with the result of Bekenstein and Mukhanov.

1.4 Quantum Black Holes as Atoms. Outlook

Since there has been no experimental evidence for either semiclassical Hawking radiation or the line spectrum of black holes, the question of whether the spectrum of horizon area is equidistant or not still does not have a definite answer, and further work to determine of the correct area quantization is necessary. In this work we will use the conjecture of Bekenstein that small black holes can be described in a similar manner as elementary particles in quantum mechanics [10]. He claimed that a black hole is fully characterised by a closed set of quantum operators $\left\{\hat{Q},\, \hat{\boldsymbol{J}}^2,\, \hat{J}_z, \hat{A}\right\}$ and some creation operators $\hat{R}_{\lambda s}$ for black holes in their various states. Using purely algebraic methods he was able to derive the algebra of the creation and area operators. However, no physical justification of the origin of either of the operators $\hat{A}$ or $\hat{R}_{\lambda s}$ was given. The operator $\hat{Q}$ classically originated as the electric charge of the black hole, whereas in quantum mechanics it is converted into the generator of $U(1)$ gauge transformations. Since general relativity is invariant under diffeomorphism transformations, the corresponding generators should also be included among the operators needed for a complete description of a black hole quantum state.

Motivated by this observation, we will investigate the properties of the algebra of the diffeomorphism constraints $\mathcal{H}^\alpha$, which arise in the covariant form of general relativity [14]. We will review the methods of the Hamiltonian formalism in chapter 2. As we will discover, the two dimensional diffeomorphism algebra of spacelike constraints can be regarded as a two dimensional extension of the Virasoro algebra. The latter is of great importance in string theory and conformal field theory(CFT) where it is the algebra of the generators of conformal transformations. The role of diffeomorphisms in string theory and CFT will be discussed in chapters 3 and 4. In chapter 5 we will present the algebraic description of black holes, suggested by Bekenstein in the light of knowledge from string quantization.

A detailed discussion of quantum extensions of two dimensional diffeomorphism algebra will be provided in chapter 6. In analogy with conformal field theory we will consider the highest-weight representation space of the diffeomorphism algebra on a closed two dimensional spacelike surface. We will consider the possibility to identify the area and creation operators in Bekenstein's description with some of

the operators present in the diffeomorphism algebra of general relativity. Summary and conclusions will be given in chapter 7.

2

Constrained Hamiltonian systems

The usual approach to describe the dynamics of a classical field theory is the action principle. The invariance of the action under some group of local symmetry transformations leads to severe restrictions on the allowed form of the Lagrangian density. This makes it possible to guess the Lagrangian even if the explicit nature of the theory is not known. The quantum theory is then derived by the approach of canonical quantization. It seems, however, that the symmetries which are apparent in the Lagrange formalism tend to "disappear" on the way to the Hamilton formalism. Moreover, the explicit distinction between spatial and time coordinates in the canonical form of the action looks rather artificial for diffeomorphism invariant theories such as general relativity. However, the local symmetries seem to also be explicit in the Hamiltonian formalism [15], which is much more suitable for quantization. The aim of this chapter is to rewrite the action of general relativity in canonical form and to derive the constraints which both generate the dynamics of general relativity and account for the diffeomorphism invariance of the Einstein action. We will, therefore, begin with a quick review of the basics of the Hamilton formalism with constraints and reveal the role of reparametrization invariance in this formalism.

2.1 The Hamilton Formalism and Constraints

2.1.1 Action in Canonical Form

Let us start with the classical action

$$S = \int dt\, L(q, \dot{q}, t), \tag{2.1}$$

where $q = \{q^1, \ldots, q^n\}$ denotes the set of generalized coordinates, and n is the number of degrees of freedom. This yields the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i}. \tag{2.2}$$

After introducing conjugated momenta,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i}, \tag{2.3}$$

and defining the Hamiltonian as

$$H(p, q) = p_i \dot{q}^i - L(q, \dot{q}, t) \tag{2.4}$$

the equations of motion become

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.5}$$

These are first order differential equations, which leads to the Hamilton formalism sometimes being referred to as the *first order formalism*. Introducing the classical Poisson bracket for some functions $f(q, p)$, $g(q, p)$ of canonical variables q and p

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} \tag{2.6}$$

enables us to rewrite equations (2.5) as Heisenberg equations

$$\dot{q}^i = \left\{q^i, H\right\}, \qquad \dot{p}_i = \{p_i, H\}. \tag{2.7}$$

The Poisson bracket of the generalized coordinates with their conjugated momenta is $\{q^i, p_j\} = \delta^i_j$. This can be straightforwardly translated into equations for quantum operators by replacing the canoncial variables q, p with non-commuting quantum operators satisfying the equal-time commutation relation $[q^i, p_j] = i\hbar\delta^i_j$. From this it follows that the classical Poisson bracket can be substituted with quantum commutator according to the rule

$$[\dots] = i\hbar\{\dots\}. \tag{2.8}$$

It thus seems that for quantizing a theory all we need is a Hamilton function.

Here we return to the question of whether the Hamiltonian reflects all the symmetries of the Lagrangian we started with. An important contribution in this direction is due to Dirac [16] who developed the quantum theory of constrained Hamiltonian systems, in which the canonical variables obey the constraint equations $\mathcal{C}_\alpha(q,p) \equiv 0$. These reflect the presence of a local symmetry in the system, and, thus, the algebra obeyed by the constraints is the algebra of the generators of the symmetry transformations. In the canonical formalism these constraints are taken into account by adding them to the Hamiltonian:

$$H_T(q,p) = H(q,p) + \mathcal{N}^\alpha\mathcal{C}_\alpha, \tag{2.9}$$

where $\mathcal{N}^\alpha = \mathcal{N}^\alpha(q,p)$ are Lagrangian multipliers.

The constraints arise, for example, in cases when the conjugated momenta p_i are not mutually independent and thus there exist several linear combinations of momenta which are zero even if the corresponding momenta themselves do not vanish. Constraints arising this way are called *primary constraints*. Dirac gave a neat example to explain the origin of such constraints. Consider a Lagrangian which is a homogeneous function of the first degree in velocities:

$$\dot{q}^i\frac{\partial L}{\partial q^i} = L.$$

From here it follows that the Hamiltonian $H = p_i\dot{q}^i - L$ is zero and thus there are no dynamics. But let us count the degrees of freedom. We started with n coordinates q^i, but, because of the specific form of Lagrangian, the conjugated momenta can only depend on the ratios of velocities. Out of n variables only $n-1$ ratios can

be built, which leaves one combination $\mathcal{C}_1(p,q)$ of q's and p's equal to zero. This can now be multiplied by an arbitrary function $\mathcal{N}^1$, and the total Hamiltonian is $H_T = \mathcal{N}^1\mathcal{C}_1$. Hence, we have included some extra information in our Hamiltonian as a direct consequence of a certain symmetry of the original Lagrangian.

2.1.2 Action in Parametrized Form

As we have seen so far, starting from a classical action principle and passing to the first order formalism we were able to obtain a quantum theory. The subtle point which we have neglected so far is whether the resulting quantum theroy is still Lorentz invariant. Although we started with a classically Lorentz invariant theory, the equations of motion in Hamiltonian form (2.7) are not manifestly covariant. The reason for this is that by referring to one absolute time, we break the four dimensional Lorentz symmetry. In order to ensure relativistic invariance, let us treat the "absolute time" t as another generalized coordinate which "evolves" as a function of some "new time" τ. Then a system with n degrees of freedom described by n coordinates q^i, becomes a system of $n+1$ degrees of freedom with $q^{n+1} = t(\tau)$. As a result the action becomes

$$\begin{aligned} S &= \int dt\, L(q, \dot{q}, t) \\ &= \int d\tau\, L^*(q, q', q^{n+1}, q^{n+1\prime}, \tau), \end{aligned} \tag{2.10}$$

where $q' \equiv dq/d\tau$. Rewriting the action in canonical form leads to

$$\begin{aligned} S &= \int d\tau \frac{dt}{d\tau}\left(\sum_{i=1}^{n} p_i \frac{dq^i}{d\tau}\frac{d\tau}{dt} - H(q,p)\right) \\ &= \int d\tau \left(\sum_{i=1}^{n} p_i q^{i\prime} - q_{n+1}{}' H(p,q)\right). \end{aligned}$$

After identifying $p_{n+1} = -H(q,p)$ the last term can be absorbed in the sum. This relation between the new conjugated momenta and the Hamiltonian can be rewritten as a constraint

$$\mathcal{C}_0(q,p) \equiv p_{n+1} + H(q,p) = 0$$

and taken into account in the action as

$$S = \int d\tau \left(\sum_{i=1}^{n+1} p_i q^{i\prime} - \mathcal{N}^0 \mathcal{C}_0(q,p) \right), \tag{2.11}$$

where q and p now denote the set of $n+1$ variables and $\mathcal{N}^0$ is the Lagrange multiplier. From here it is obvious that we now have obtained the reparametrization invariance of the "time" variable $\tau \to \tilde{\tau}(\tau)$. This will give an extra factor $d\tau/d\tilde{\tau}$ in front of the Lagrange multiplier $\mathcal{N}^0$. But, as long as $\mathcal{N}^0 = \mathcal{N}^0(\tau)$ is an arbitrary function of the parameter τ only, the "time" reparametrization corresponds to a trivial redefinition of the Lagrange multiplier $\tilde{\mathcal{N}}^0(\tilde{\tau}) = \frac{d\tau}{d\tilde{\tau}}\mathcal{N}^0(\tau)$. Thus, we have rewritten the action (2.10) in a manifestly covariant form.

In the case that there are m extra constraints like, for example, the primary constraints introduced in the previous section they can also be taken into account in the action:

$$S = \int d\tau \left(\sum_{i=1}^{n+1} p_i q^{i\prime} - \sum_{\alpha=0}^{m} \mathcal{N}^\alpha \mathcal{C}_\alpha \right). \tag{2.12}$$

This is called *the action in parametrized form*. Note that the Hamilton function of a theory whose action is expressed in parametrized form, according to (2.4), is just a combination of constraints

$$H_T(q,p) = \sum_{\alpha=0}^{m} \mathcal{N}^\alpha \mathcal{C}_\alpha(q,p).$$

In such a case we say that the Hamilton function is *weakly* zero. The resulting equation of motion for a general function of dynamical variables, $g(q,p)$, is then

$$\frac{dg(q,p)}{d\tau} \approx \left\{ g, \sum_{\alpha=0}^{m} \mathcal{N}^\alpha \mathcal{C}_\alpha(q,p) \right\}. \tag{2.13}$$

The curly equality sign means that the constraints have to be set to zero *after* the equation of motion is solved. The reparametrization $\tau \to \tau'(\tau)$ leaves the equation of motion unaffected and thus it is obvious that the resulting theory is now covariant.

2.2 The Covariant Form of General Relativity

2.2.1 Diffeomorphism Invariance of General Relativity

The Einstein-Hilbert action for the gravitational field is

$$S = -\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R. \tag{2.14}$$

The equations of motion obtained by varying the action with respect to the metric $g^{\alpha\beta}$ are

$$\delta S = \int d^4x\,\frac{\delta S}{\delta g^{\alpha\beta}}(x)\delta g^{\alpha\beta}(x) = -\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R)\delta g^{\alpha\beta}. \tag{2.15}$$

By setting this variation to zero one obtains ten equations for the seemingly independent components of the metric:

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0, \tag{2.16}$$

where $G_{\alpha\beta}$ is the symmetric Einstein tensor. These equations are highly non-linear and impossible to solve for the general case. The theory of general relativity is manifestly invariant under the general coordinate transformation $x^\mu \to \tilde{x}^\mu(x^\nu)$. The physical origin of this invariance is clear, as the change of the spacetime coordinate system simply corresponds to "renaming the points" of the manifold, which consists of events. It is clear that this does not change the physics, just as "renaming of the streets" does not change the buildings in the city. Hence, the local symmetry of the general relativity Lagrangian is its invariance under infinitesimal local diffeomorphism transformations:

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x). \tag{2.17}$$

A very important feature of general relativity is that the equations of motion for matter do not need to be postulated separately, but follow from the Bianchi identities, $G^\alpha_{\beta;\alpha} = 0$, satisfied by the Einstein tensor. The Bianchi identities can be derived explicitly from the properties of the Riemann tensor, but they also follow from the diffeomorphism invariance of the Lagrangian of general relativity. To see this consider the infinitesimal transformation law of the metric under the

transformation (2.17):

$$g^{\alpha\beta}(x) \to \tilde{g}^{\alpha\beta}(x) = g^{\alpha\beta}(x) - g^{\alpha\beta},_{\lambda}\,\xi^{\lambda} + g^{\beta\lambda}\xi^{;\alpha}_{\lambda} + g^{\alpha\lambda}\xi^{;\beta}_{\lambda}, \tag{2.18}$$

$$\Rightarrow \qquad \delta g^{\alpha\beta} = \xi^{\alpha;\beta} + \xi^{\beta;\alpha}. \tag{2.19}$$

Note that the argument x is the same on both sides. Thus we compare the metric at different points of the manifold, which have the same coordinate values in both coordinate systems $\{x^{\mu}\}$ and $\{\tilde{x}^{\mu}\}$. Under this transformation the action changes as

$$\begin{aligned}
\delta S &= -\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}G_{\alpha\beta}\delta g^{\alpha\beta} \\
&= -\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}G_{\alpha\beta}(\xi^{\alpha;\beta} + \xi^{\beta;\alpha}) \\
&= -\frac{1}{8\pi G}\int d^4x\,\sqrt{-g}G^{;\beta}_{\alpha\beta}\xi^{\alpha} \\
&= 0,
\end{aligned} \tag{2.20}$$

and from this it follows that $G^{\alpha}_{\beta;\alpha} = 0$. Hence, we have derived the Bianchi identities by exploiting the invariance of the action under diffeomorphisms, and without explicitly referring to the properties of Ricci scalar.

2.2.2 The Hamilton Formalism

As we have seen, one consequence of the invariance of general relativity under general coordinate transformations is that the number of independent components of metric is reduced from ten to six. Hence, there are only six dynamical variables in general relativity, and these will appear with first order time derivatives in the canonical form of the action. To rewrite the Lagrangian of general relativity in the first order form we use the Hilbert-Palatini formalism. In this one treats the metric and the connection as independent variables and thus the Lagrangian is linear in first derivatives of g and Γ.

We will begin with the action

$$S = \int d^4x\,\sqrt{-g}R \tag{2.21}$$

where we adapt the sign and units conventions of [14]. Furthermore, we will use Planck units throughout the rest of our work. In the spirit of previous chapter we see that the action is written in an already reparametrization invariant form with no particular role associated to the time coordinate. This implies that the total Hamiltonian of general relativity vanishes weakly and can be expressed as linear combination of the constraints, which reflect the diffeomorphism invariance of the theory. There are four allowed diffeomorphism transformations associated with each spacetime direction, and, hence, we expect four constraints arising on the way to the Hamilton formalism. In order to find these we will exploit gauge freedom to choose a particularly convenient coordinate system.

2.2.2.1 Splitting the Spacetime in 3+1

Let us slice the spacetime into a one-parameter family of spacelike hypersurfaces. The use of this specific spacetime decomposition does not, of course, impair the general invariance of the theory under arbitrary coordinate transformations. Using this splitting we will rewrite the action of general relativity in parametrized form (2.12), where the Hamiltonian and time parameter are again introduced as a conjugated pair of generalized coordinates.

Consider two such subsequent spacelike hypersurfaces Σ_t and Σ_{t+dt} with $t =$ const and $t + dt =$ const respectively. The geometry of the "earlier" hypersurface is described by the 3-dimensional metric

$$\gamma_{ij}(t, x, y, z)dx^i dx^j;$$

the metric on the "later" hypersurface is

$$\gamma_{ij}(t + dt, x, y, z)dx^i dx^j.$$

In order to fix the geometry of the spacetime one has to specify the rules by which the points on different equal-time slices are connected. This enables one to calculate the proper interval ds^2 between two spacetime points $x^\mu = (t, x^i)$ and $x^\mu + dx^\mu = (t + dt, x^i + dx^i)$ by using the Pythagorean theorem:

$$ds^2 = \gamma_{ij}(dx^i + \mathcal{N}^i dt)(dx^j + \mathcal{N}^j dt) - (\mathcal{N}^0 dt)^2.$$

This yields to 3+1 decomposition of the metric tensor

$$(g_{\alpha\beta}) = \begin{pmatrix} -\mathcal{N}^2 + \mathcal{N}_i\mathcal{N}^i & \mathcal{N}_i \\ \mathcal{N}_i & \gamma_{ij} \end{pmatrix} \tag{2.22}$$

The covariant lapse and shift variables are given by

$$\mathcal{N}^i = \gamma^{ij}\mathcal{N}_j, \qquad \mathcal{N}^0 = \mathcal{N}_0 = \mathcal{N}$$

and the inverse of the metric is

$$\left(g^{\alpha\beta}\right) = \frac{1}{\mathcal{N}^2}\begin{pmatrix} -1 & \mathcal{N}^i \\ \mathcal{N}^i & \mathcal{N}^2\gamma^{ij} - \mathcal{N}^i\mathcal{N}^j \end{pmatrix}. \tag{2.23}$$

For the proper volume element the determinant will be needed

$$g = \det(g_{\alpha\beta}) = -\mathcal{N}^2\gamma, \qquad \text{with} \qquad \gamma \equiv \det(\gamma_{ij}).$$

2.2.2.2 Constraints in ADM Formalism

In this section we are going to rewrite the action (2.21) of general relativity in the first order form. When we say *first order* we mean that we are looking for a form in which the generalized coordinates and momentum appear in the Lagrangian in the combination $p\dot{q}$, i.e. with *first* time derivatives. After varying the action with respect to p and q one obtains *first* order equations of motion. Furthermore, we will use Hilbert-Palatini method and treat the quantities g and Γ independently. The end result for the Lagrangian density is:

$$\mathcal{L} = \pi^{ij}\dot{\gamma}_{ij} - \mathcal{N}^\alpha\mathcal{H}_\alpha, \tag{2.24}$$

where π^{ij} are momenta conjugated to γ_{ij} and defined in terms of the extrinsic curvature K_{ij},

$$\begin{aligned} \pi^{ij} &= -\gamma^{1/2}\left(K^{ij} - \gamma^{ij}K\right) \\ K_{ij} &= \frac{1}{2}\mathcal{N}^{-1}\left(\mathcal{N}_{i;j} + \mathcal{N}_{j;i} - \gamma_{ij,0}\right), \qquad K = \gamma^{ij}K_{ij}, \end{aligned}$$

and $\mathcal{N}^\alpha$ are the lapse and shift variables introduced above. $\mathcal{H}_\alpha$ are the constraints due to the diffeomorphism invariance of general relativity

$$\mathcal{H}_0 = G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{\gamma}^{(3)}R \tag{2.25}$$

$$\mathcal{H}_i = -2\gamma_{ij}\pi^{jl}_{|l}, \tag{2.26}$$

where

$$G_{ijkl} = \frac{1}{2\sqrt{\gamma}}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}).$$

${}^{(3)}R$ is the intrinsic curvature of a hypersurface of constant time t and the vertical bar denotes the covariant derivative with respect to the 3-metric γ_{ij}. After using the generalized Poisson brackets

$$\left\{\gamma_{ij}(x), \pi^{kl}(y)\right\} = \delta_i^{(k}\delta_j^{l)}\delta(x,y) = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta(x,y), \tag{2.27}$$

where x and y denote two different points on the spacetime manifold, we can derive the following equal-time Poisson brackets for the constraints

$$\begin{aligned}
\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} &= \gamma^{ij}(x)\mathcal{H}_j(x)\frac{\partial}{\partial x^i}\delta(x,y) - \gamma^{ij}(y)\mathcal{H}_j(y)\frac{\partial}{\partial y^i}\delta(x,y), \\
\{\mathcal{H}_i(x), \mathcal{H}_0(y)\} &= \mathcal{H}_0(x)\frac{\partial}{\partial x^i}\delta(x,y), \\
\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} &= \mathcal{H}_j(x)\frac{\partial}{\partial x^i}\delta(x,y) - \mathcal{H}_i(y)\frac{\partial}{\partial y^j}\delta(x,y).
\end{aligned} \tag{2.28}$$

These then form a closed set of constraints, i.e. in the language of Dirac they are *first class* constraints. The $\mathcal{H}_0$ constraint, being the generator of translations in the time direction, describes the time evolution of the gravitational field, while $\mathcal{H}_i$ generate diffeomorphism transformations on the hypersurface Σ_t.

3

Diffeomorphisms and Physical Quantum States in String Theory

In this chapter the role of the diffeomorphism invariance in string theory is investigated. We will begin with the classical theory and will show how the diffeomorphism constraints arise in both the Lagrange and Hamilton formalisms. We then consider the canonical and light-cone gauge quantization of string theory and explore how the classical constraints are resolved in these approaches. In the conclusion an explicit construction of the physical quantum string state space by the use of vertex operators is presented. This chapter follows the books [17; 18].

3.1 Symmetries of the Polyakov Action

Consider a free bosonic string. Its trajectory in the target spacetime covers a two dimensional hypersurface called a *worldsheet*. We parametrize this hypersurface by one timelike coordinate τ and one spacelike coordinate σ taking values in the range $\sigma \in [0, 2\pi]$. The worldsheet coordinates (τ, σ) are mapped to target space coordinates $X^\mu(\sigma, \tau)$, where $\mu = 0, ..., D-1$. Target space is assumed to be D-dimensional, flat Minkowski space with the metric $\eta_{\mu\nu} = (-1, 1, \ldots, 1)$. The action for the string has to be proportional to the area of the worldsheet. This is a two-dimensional generalization of the action for a relativistic particle moving along geodesics. Instead of minimizing the length of the worldline, the string minimizes the area of its worldsheet.

3. DIFFEOMORPHISMS AND PHYSICAL QUANTUM STATES IN STRING THEORY

In practice it is more convenient to work with the Polyakov action

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}(\sigma,\tau)\eta_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu, \tag{3.1}$$

Here $g_{\alpha\beta}(\sigma)$ is the metric on the string worldsheet. Since no time derivatives of the metric appear in the Lagrangian, the equations of motion for $g_{\alpha\beta}$ are the constraints in string theory. After imposing these constraints, the action reduces to the area of the worldsheet. The proportionality constant $T = \frac{1}{2\pi\alpha'}$ is a parameter of dimension $[L]^{-2}$ and has a physical interpretation as the tension of the string, i.e. potential energy per unit length. α' is a conventional parameter called the Regge slope parameter. The indices that α, β take values $0, 1$ while $\mu, \nu = 0, ..., D-1$.

The Polyakov action (3.1) is invariant under both global, D-dimensional, Poincaré transformations and local diffeomorphism transformations on the string worldsheet. This action can also be interpreted as describing D scalar fields $X^\mu(\tau,\sigma)$ in a curved two-dimensional spacetime. The invariance under general coordinate transformations enables us, by appropriate choice of gauge, to bring the worldsheet metric to a conformally flat form:

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \mathrm{e}^{2\Phi}\eta_{\alpha\beta}.$$

This choice of worldsheet metric is referred to as conformal gauge. Moreover, the action is also locally Weyl invariant, i.e. the transformation

$$g_{\alpha\beta}(\tau,\sigma) \to \Omega^2(\tau,\sigma)g_{\alpha\beta}(\tau,\sigma)$$

leaves it unchanged. Hence the conformal factor $e^{2\Phi}$ drops out of the Polyakov action which in conformal gauge becomes

$$S = -\frac{T}{2}\int d^2\sigma\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu. \tag{3.2}$$

However, some reparametrization freedom is still left because requiring that $ds^2 = e^{2\Phi}(d\tau^2 - d\sigma^2)$ does not uniquely fix the coordinate system. In fact, in the worldsheet light-cone coordinates

$$\sigma^\pm = \tau \pm \sigma, \qquad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$$

the line element becomes $ds^2 = e^{2\Phi} d\sigma^+ d\sigma^-$. Under conformal transformations $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$ it transforms to $ds^2 = e^{2\tilde{\Phi}} d\tilde{\sigma}^+ d\tilde{\sigma}^-$. The prefactor has changed, but the metric is still conformally flat. This residual symmetry plays an important role in string theory. I will consider it in detail in section 3.5.1.

In conformal gauge the equations of motion for X^μ are

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{3.3}$$

As we have assumed that the worldsheet metric is an independent field, then the equation of motion for $g_{\alpha\beta}$ also has to be satisfied. Recalling the definition of the energy-momentum tensor

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} = 0,$$

it turns out that satisfying the equations of motion for $g_{\alpha\beta}$ classically corresponds to setting $T_{\alpha\beta} = 0$. These are the constraints of string theory. In conformal gauge the constraint equations are

$$C_0 \equiv T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0, \tag{3.4}$$

$$C_1 \equiv T_{01} = T_{10} = \dot{X} \cdot X' = 0. \tag{3.5}$$

Here $\dot{X} \equiv \frac{\partial X}{\partial \tau}$, $X' \equiv \frac{\partial X}{\partial \sigma}$, and the scalar product is denoted as $X \cdot X = \eta_{\mu\nu} X^\mu X^\nu$.

3.2 The Canonical Form of the Polyakov Action

In this section we will treat string theory as a theory for D massless scalar fields on a two dimensional background with metric $g_{\alpha\beta}$. Instead of choosing conformal gauge, we will use the 1+1 decomposition for the metric $g_{\alpha\beta}$:

$$ds^2 = -(\mathcal{N}^2 - \mathcal{N}^1\mathcal{N}_1)d\tau^2 + 2\mathcal{N}_1 d\sigma d\tau + \gamma_{11} d\sigma^2, \tag{3.6}$$

where $\mathcal{N}$ and $\mathcal{N}_1$ are the lapse and shift respectively. After introducing the momenta conjugated to the scalar field X^μ:

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \frac{\sqrt{\gamma_{11}}}{\mathcal{N}}(\dot{X}_\mu - \mathcal{N}^1 X_\mu{}'), \tag{3.7}$$

the Polyakov action (3.1) in the first order Hamilton formalism takes the following form

$$S = \int d^2\sigma \, (\pi_\mu \dot{X}^\mu - \mathcal{N}^\alpha C_\alpha), \tag{3.8}$$

where $T \equiv 1$, $\mathcal{N}^0 = \frac{\mathcal{N}}{\sqrt{\gamma}}$ and $\mathcal{N}^1$ the lapse and the shift are Lagrange multipliers and the constraints are

$$C_0 = \frac{1}{2}(\pi^2 + X'^2), \qquad C_1 = \pi_\mu X'^\mu. \tag{3.9}$$

Note that in the conformal gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ the conjugated momenta in (3.7) reduce to $\pi_\mu = \dot{X}_\mu$ and the constraints reduce to (3.4) and (3.5). The following equal-time Poisson brackets for the constraints can be derived as

$$\begin{aligned} \{C_i(\sigma), C_i(\sigma')\} &= C_1(\sigma)\frac{\partial}{\partial\sigma}\delta(\sigma,\sigma') - C_1(\sigma')\frac{\partial}{\partial\sigma'}\delta(\sigma,\sigma'), \\ \{C_0(\sigma), C_1(\sigma')\} &= C_0(\sigma)\frac{\partial}{\partial\sigma}\delta(\sigma,\sigma') - C_0(\sigma')\frac{\partial}{\partial\sigma'}\delta(\sigma,\sigma') \end{aligned} \tag{3.10}$$

with $i = 0, 1$. These constraints are consequences of the diffeomorphism invariance of the Polyakov action.

3.3 Mode Expansions

3.3.1 Constraints in Light-cone Worldsheet Coordinates

Consider a closed string which obeys the periodicity condition

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi).$$

In terms of the worldsheet light-cone coordinates equations of motion (3.3) can be written as

$$\partial_+\partial_- X^\mu = 0. \tag{3.11}$$

The general solution for these equations can be written as a sum of left- and right-moving modes

$$X^\mu(\tau, \sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-).$$

The most general solution of equation (3.11) is then the following expansion in Fourier series (the coefficients are adjusted to agree with common notations)

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu \mathrm{e}^{-in\sigma^+},$$
$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \mathrm{e}^{-in\sigma^-}, \tag{3.12}$$

where x^μ and p^μ are the position and momenta of the center of mass of the string. By demanding that $\pi^\mu = \dot{X}^\mu$ and X^μ obey the canonical Poisson bracket one can show that the Fourier coefficients α_n^μ have the following Poisson brackets

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = im\delta_{m+n}\eta^{\mu\nu} \tag{3.13}$$
$$\{\alpha_m^\mu, \tilde{\alpha}_n^\nu\} = 0.$$

The requirement that X_R and X_L are real functions leads to further restrictions on the Fourier components

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger, \quad \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^\dagger.$$

For later use and reference let us write down the expressions for $X^\mu(\tau,\sigma)$ and its derivatives explicitly:

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in\sigma^-} + \tilde{\alpha}_n^\mu e^{-in\sigma^+}\right), \tag{3.14}$$
$$\dot{X}^\mu(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}}\left(\sum_n \tilde{\alpha}_n^\mu e^{-in\sigma^+} + \sum_n \alpha_n^\mu e^{-in\sigma^-}\right), \tag{3.15}$$
$$X^{\mu\prime}(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}}\left(\sum_n \tilde{\alpha}_n^\mu e^{-in\sigma^+} - \sum_n \alpha_n^\mu e^{-in\sigma^-}\right). \tag{3.16}$$

where $\alpha_0^\mu \equiv \tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu$. The constraints (3.4) in light-cone coordinates become

$$T_{+-} = T_{-+} = 0, \tag{3.17}$$

$$L(\tau,\sigma) \equiv T_{--} = \frac{1}{2}(T_{00} - T_{01}) = (\partial_- X)^2 = \frac{\alpha'}{2} \sum_{m,n} \alpha_{n-m} \cdot \alpha_m \mathrm{e}^{-in\sigma^-} = 0, \tag{3.18}$$

$$\bar{L}(\tau,\sigma) \equiv T_{++} = \frac{1}{2}(T_{00} + T_{01}) = (\partial_+ X)^2 = \frac{\alpha'}{2} \sum_{m,n} \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m \mathrm{e}^{-in\sigma^-} = 0.$$

Hence the light-cone constraints L, $\bar{L}$ are related to "Minkowski worldsheet" constraints C_0, C_1 as

$$\begin{aligned} L &= \frac{1}{2}(C_0 - C_1) \\ \bar{L} &= \frac{1}{2}(C_0 + C_1). \end{aligned} \tag{3.19}$$

The above expansions can be written in a shorter form

$$\begin{aligned} L(\tau,\sigma) &= \alpha' \sum_n L_n \mathrm{e}^{-in\sigma^-} = 0, \\ \bar{L}(\tau,\sigma) &= \alpha' \sum_n \bar{L}_n \mathrm{e}^{-in\sigma^+} = 0 \end{aligned} \tag{3.20}$$

Here we have introduced the Virasoro modes L_n, $\bar{L}_n$. They are the Fourier coefficients of the above expansion, evaluated at time $\tau = 0$,

$$\begin{aligned} L_n &= \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \mathrm{e}^{-in\sigma} L(0,\sigma), \\ \bar{L}_n &= \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \mathrm{e}^{in\sigma} \bar{L}(0,\sigma). \end{aligned} \tag{3.21}$$

The Poisson brackets for L_n and $\bar{L}_n$ can be calculated directly from definitions (3.21) and Poisson brackets (3.13). This yields the Virasoro algebra

$$\begin{aligned} \{L_n, L_m\} &= -i(n-m)L_{m+n}, \\ \{\bar{L}_n, \bar{L}_m\} &= -i(n-m)\bar{L}_{m+n}, \\ \{L_n, \bar{L}_m\} &= 0. \end{aligned} \tag{3.22}$$

As the Virasoro operators L_n and $\bar{L}_n$ decouple, we will often consider only one copy of these algebras.

The constraint equations (3.17) in terms of the Virasoro modes become:

$$L_n = \bar{L}_n = 0, \qquad \forall n.$$

3.3.2 Constraints in Minkowski Worldsheet Coordinates

We can similarly expand the constraints (3.4) in a Fourier series on the circle as

$$C_i(0,\sigma) = \sum_{n=-\infty}^{+\infty} (C_i)_n e^{in\sigma}, \qquad (C_i)_n = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, C_i(0,\sigma) e^{-in\sigma}. \tag{3.23}$$

From the Poisson brackets (3.10) it follows that the Fourier modes $(C_i)_n$ also obey the Virasoro algebra

$$\begin{aligned} \{(C_i)_n, (C_i)_m\} &= -i(n-m)(C_i)_{n+m}, \\ \{(C_0)_n, (C_1)_m\} &= -i(n-m)(C_0)_{n+m}. \end{aligned} \tag{3.24}$$

In distinction from the light-cone constraint algebra (3.22), the modes $(C_{0,1})_n$ do not decouple. In terms of string oscillators the Fourier coefficients of C_1, C_0 at $\tau = 0$ can be expressed as

$$\begin{aligned} (C_0)_n &= \frac{1}{2}\sum_m (\tilde{\alpha}_{-n-m}\cdot\tilde{\alpha}_m + \alpha_{n-m}\cdot\alpha_m), \\ (C_1)_n &= \frac{1}{2}\sum_m (\tilde{\alpha}_{-n-m}\cdot\tilde{\alpha}_m - \alpha_{n-m}\cdot\alpha_m). \end{aligned} \tag{3.25}$$

Further we observe that the Fourier modes L_n, $\bar{L}_n$ and $(C_0)_n$, $(C_1)_n$ are <u>not</u> related as in eq. (3.19). Instead they satisfy

$$\begin{aligned} (C_0)_n &= \bar{L}_n^\dagger + L_n, \\ (C_1)_n &= \bar{L}_n^\dagger - L_n. \end{aligned}$$

This is due to the differences in the Fourier expansions (3.21) and (3.24). However, the constraints C_0, C_1 are both more natural and easier to interpret because the distinction between timelike and spacelike coordinates is preserved.

3.3.3 Generators of Diffeomorphism Transformations

There is a straightforward interpretation of both sets of Virasoro operators L_n, $\bar{L}_n$ and $(C_0)_n$, $(C_1)_n$. This can be clearly seen by calculating the action of the generators L_n, $\bar{L}_n$ on the target space coordinates X^μ:

$$\{L_n, X^\mu(\tau,\sigma)\} = e^{in\sigma^-}\partial_- X^\mu \tag{3.26}$$

$$\left\{\bar{L}_n, X^\mu(\tau,\sigma)\right\} = e^{in\sigma^+}\partial_+ X^\mu \tag{3.27}$$

Hence, the Virasoro operators L_n ($\bar{L}_n$) generate the residual diffeomorphism transformations which preserve the conformal gauge. As σ^- (σ^+) is an angular variable in the mode expansion of X_L (X_R) then it can also be said that L_n ($\bar{L}_n$) generate diffeomorphisms on the circle.

Similarly, for the equal time constraints $(C_0)_n$, $(C_1)_n$ we have

$$\{(C_0)_n, X^\mu(\tau,\sigma)\} = e^{-in\sigma}\left(\dot{X}^\mu \cos n\tau - iX'^\mu \sin n\tau\right) \tag{3.28}$$

$$\{(C_1)_n, X^\mu(\tau,\sigma)\} = e^{-in\sigma}\left(-i\dot{X}^\mu \sin n\tau + X'^\mu \cos n\tau\right). \tag{3.29}$$

Here the time and space directions of the worldsheet are mixed and the interpretation might seem confusing. In order to clarify this, let us, instead of choosing a given moment of time $\tau = 0$, consider a constant spatial coordinate $\sigma = 0$. Then the constraints C_0 and C_1 become

$$C_0(\tau, \sigma = 0) = \frac{\alpha'}{2}\sum_{n,m}\left(\tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m + \alpha_{n-m}\cdot\alpha_m\right)e^{-in\tau}, \tag{3.30}$$

$$C_1(\tau, \sigma = 0) = \frac{\alpha'}{2}\sum_{n,m}\left(\tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m - \alpha_{n-m}\cdot\alpha_m\right)e^{-in\tau}.$$

We then define the corresponding Fourier coefficients as

$$(A_0)_n \equiv \frac{1}{2}\sum_m\left(\tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m + \alpha_{n-m}\cdot\alpha_m\right), \tag{3.31}$$

$$(A_1)_n \equiv \frac{1}{2}\sum_m\left(\tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m - \alpha_{n-m}\cdot\alpha_m\right). \tag{3.32}$$

In order to calculate the Poisson bracket for two fields, which are given at differ-

ent moments of time, in generic case, one should use the time evolution operator determined by the Hamiltonian, to "bring the fields" to equal times. However, one can formally use the expressions (3.30) and Poisson brackets (3.13) to derive the following relations:

$$\{C_i(\tau), C_i(\tau')\} = -\left(C_0(\tau)\frac{\partial}{\partial\tau}\delta(\tau,\tau') - C_0(\tau')\frac{\partial}{\partial\tau'}\delta(\tau,\tau')\right), \tag{3.33}$$

$$\{C_0(\tau), C_1(\tau')\} = -\left(C_1(\tau)\frac{\partial}{\partial\tau}\delta(\tau,\tau') - C_1(\tau')\frac{\partial}{\partial\tau'}\delta(\tau,\tau')\right). \tag{3.34}$$

Note that these equations are very similar to the Poisson brackets (3.10). One should be aware, however, that the equal-time Poisson brackets are universal, i.e. they do not depend on the equations of motion. Meanwhile, the "equal-space Poisson brackets" were derived by explicitly using the solution of the equations of motion. However, the solutions of string theory are mode expanded in the light-cone worldsheet coordinates $\sigma^{\pm} = \tau \pm \sigma$. Hence, the space and time worldsheet coordinates appear in combinations of the light-cone coordinates only. It follows then from the remaining gauge invariance of Polyakov action that under the conformal transformation $\sigma^- \to -\sigma^-$ the roles of the coordinates τ and σ are exchanged. Therefore, the equal-time Poisson brackets have the same structure as the "equal-space Poisson brackets".

The action of $(A_0)_n$ and $(A_1)_n$ on coordinates X^μ is

$$\{(A_0)_n, X^\mu(\tau,\sigma)\} = e^{in\tau}\left(\dot{X}^\mu \cos n\sigma + iX^{\mu\prime}\sin n\sigma\right), \tag{3.35}$$

$$\{(A_1)_n, X^\mu(\tau,\sigma)\} = e^{in\tau}\left(i\dot{X}^\mu \sin n\sigma + X^{\mu\prime}\cos n\sigma\right), \tag{3.36}$$

which coincides with (3.28) if one substitutes $\sigma = -\tau$ and $(A_1)_n = -(C_0)_n$. After rewriting

$$\begin{cases} \{(C_0)_n, X^\mu(0,\sigma)\} = e^{-in\sigma}\dot{X}^\mu \\ \{(C_1)_n, X^\mu(0,\sigma)\} = e^{-in\sigma}X^{\mu\prime}, \end{cases} \qquad \begin{cases} \{(A_0)_n, X^\mu(\tau,0)\} = e^{in\tau}\dot{X}^\mu \\ \{(A_1)_n, X^\mu(\tau,0)\} = e^{in\tau}X^{\mu\prime}. \end{cases} \tag{3.37}$$

the physical interpretation of C_0 and C_1 is obvious. Hence, the C_0 constraint generates time translations and the C_1 constraint generates spatial diffeomorphisms.

3.4 Old Canonical Quantization

As we have seen so far - the Polyakov action, initially invariant under diffeomorphism and Weyl transformations, was simplified by choosing the conformal gauge. Still, there is a residual symmetry left which does not affect the choice of conformal gauge. More precisely, the action is still invariant under conformal transformations $\sigma^{\pm} \to \tilde{\sigma}^{\pm}(\sigma^{\pm})$. This gauge symmetry was a direct consequence of promoting the worldsheet metric $g_{\alpha\beta}$ to a dynamical variable. This resulted in additional constraint equations which still have to be imposed once the solutions of the equations of motion for target space coordinates X^{μ} are found. In the worldsheet light-cone coordinates the final form of the subsidiary conditions was $L_n = \bar{L}_n = 0$.

How can we quantize the dynamical degrees of freedom of the string? The Gupta-Bleuler method known from quantum electrodynamics suggests that the system should be first quantized cannonically and that afterwards the constraint equations in the form of operator equations on our wave functions have to be imposed. So let us promote the target space coordinates X^{μ} to operator valued fields and define their conjugated momenta as $\pi^{\mu} = \frac{1}{2\pi\alpha'}\dot{X}^{\mu}$. We further demand that they obey the canonical equal-time commutation relations

$$\begin{aligned} &[X^{\mu}(\sigma), \pi^{\nu}(\sigma')] = i\delta(\sigma - \sigma')\eta^{\mu\nu} \\ \Rightarrow &[\alpha_n^{\mu}, \alpha_m^{\nu}] = [\tilde{\alpha}_n^{\mu}, \tilde{\alpha}_m^{\nu}] = n\eta^{\mu\nu}\delta_{n+m,0}, \quad [x^{\mu}, p^{\nu}] = i\eta^{\mu\nu}. \end{aligned} \tag{3.38}$$

After redefining $a_n = \frac{\alpha_n}{\sqrt{n}}$ and $a_n^{\dagger} = \frac{\alpha_{-n}}{\sqrt{n}}$ with $n > 0$, one obtains the familiar commutation relations for the harmonic oscillator

$$\left[a_n^{\mu}, a_m^{\nu\dagger}\right] = \delta_{n,m}\eta^{\mu\nu}, \qquad [a_n, a_m] = \left[a_n^{\dagger}, a_m^{\dagger}\right] = 0. \tag{3.39}$$

This suggests that we interpret $a_n^{\dagger}$ and a_n as creation and annihilation operators respectively. The redefinition we performed above was only useful to see clearly that α_n^{μ} and α_{-n}^{μ} can be associated with some kind of creators and annihilators and can be further used to build a Fock space. The corresponding analysis is valid also for $\tilde{\alpha}$. Thus every scalar field $X^{\mu}(\sigma)$ gives rise to an infinite number of creation and annihilation operators.

Basing on this analogy we define the vacuum state of the Fock space as

$$\alpha_n^\mu |0\rangle = \tilde{\alpha}_n^\mu |0\rangle = 0, \qquad n > 0. \tag{3.40}$$

We should be aware, however, that there is one more degree of freedom arising from the zero mode of the oscillators α_0^μ, $\tilde{\alpha}_0^\mu$. This is p^μ, which corresponds to the momenta of the center of mass of the string. Hence, we denote $|0;p\rangle$ as a state which is annihilated by the oscillators α_n^μ, $\tilde{\alpha}_n^\mu$, $n > 0$ and has a center of mass momentum p^μ. We will further restrict the discussion to one of the sets of oscillator modes only. However, we keep in mind that $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$. This relation translates into the *level matching condition* $L_0 = \bar{L}_0$ for closed strings.

Now we can build excited states by acting on the vacuum state with creation operators. Every state in the Fock space can be schematically written as

$$|\lambda\rangle = \prod_{n=1}^{\infty} \prod_{\mu=0}^{D-1} (\alpha_{-n}^\mu)^{\mu_n} |0;p\rangle \,. \tag{3.41}$$

The Fock space defined above cannot be the physical Hilbert space though. This can be seen immediately after considering the state $\alpha_{-n}^0 |0\rangle$. In order to calculate its norm the commutation relations (3.39) for the time component have to be used. As a result this state has negative norm $\langle 0| \alpha_n^0 \alpha_{-n}^0 |0\rangle = -n$. But we know that the physical space should not contain any ghosts. In order to resolve this problem, one has to implement the Virasoro constraints obtained in the classical theory, $L_n = 0$, $\forall n$, in the quantum theory.

First, let us see, how these classical modes translate into quantum operators. Because of the normal ordering of creation and annihilation operators, two modifications of the Virasoro algebra have to be made. The only normal ordering ambiguities arise in the zero mode L_0, because α_{n-m} commutes with α_m unless $n = 0$. Thus some unknown ordering constant a will appear. One chooses to define the quantum operator L_0 to be the normal ordered expression

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \tag{3.42}$$

and to include the normal ordering constant a by replacing $L_0 \to L_0 - a$ everywhere. For the same reason an extra term in the commutation relations $[L_n, L_{-n}]$ appears.

It is determined by demanding that the Jacobi identity is satisfied. The quantum Virasoro algebra is then

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n,0}. \tag{3.43}$$

In quantum theory the physicality conditions have to be implied as operator equations on the states $L_n |\phi\rangle = 0$, $\forall n$. This condition is too strong, because it leaves us with an empty Hilbert space, and must be replaced by a weaker requirement similarly to Gupta-Bleuler quantization. Namely, we demand that every physical state is annihilated by positive frequency modes only. Then by choosing normal ordering convention as 'negative frequency modes on the left and positive frequency modes on the right,' every matrix element between two physical states vanishes. Thus, the physicality conditions for a quantum state $|\phi\rangle$ read:

$$L_n |\phi\rangle = 0, \quad \forall\, n > 0 \tag{3.44}$$

$$(L_0 - a) |\phi\rangle = 0. \tag{3.45}$$

Classically ($a = 0$) the condition $L_0 = 0$ translates into a relation between the mass squared and the oscillator modes of the closed string on the mass shell:

$$M^2 = -p_\mu p^\mu = -\frac{2\alpha_0^2}{\alpha'} = -\frac{4}{\alpha'}\left(\frac{1}{2}\alpha_0^2\right) = \frac{4}{\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n. \tag{3.46}$$

Thus, there is a natural choice for the quantum mass squared operator:

$$M^2 = \frac{4}{\alpha'}\left(-a + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n\right). \tag{3.47}$$

After introducing the number operator $N \equiv \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n$, we have

$$\begin{aligned} M^2 &= \frac{4}{\alpha'}(-a + N), \\ L_0 &= \frac{1}{2}\alpha_0^2 + N. \end{aligned} \tag{3.48}$$

The commutation relations with the creation and annihilation operators are:

$$[N, \alpha_n] = -n\alpha_n, \qquad [N, \alpha_{-n}] = n\alpha_{-n}, \qquad \forall n > 0. \tag{3.49}$$

When the number operator acts on the basis state $|\lambda\rangle$ in eq. (3.41), its eigenvalue is the sum of the mode numbers of the creation operators

$$N|\lambda\rangle = N_\lambda |\lambda\rangle, \qquad \text{with} \qquad N_\lambda = \sum_{n=1}^{\infty}\sum_{\mu=0}^{25} n\mu_n. \tag{3.50}$$

Therefore, the eigenstates $|\lambda\rangle$ can be classified according to their eigenvalues N_λ. It is then said that the state $|\lambda\rangle$ is of the level N_λ.

3.5 Light-cone Gauge Quantization

The idea behind this approach to quantizing the string is to eliminate the non-dynamical degrees of freedom before passing to quantum mechanics. This is done by exploiting the remaining symmetry of the Polyakov action so that the constraint equations can be resolved at the classical level. This envolves a choice of a particular gauge, which appears to break the Lorentz invariance. However, it can be shown that by appropriate choice of normal ordering constant a and spacetime dimension D, the Lorentz invariance can be preserved.

3.5.1 Residual Gauge Symmetry

As was mentioned several times before, even after fixing $g_{\alpha\beta} = e^{2\Phi}\eta_{\alpha\beta}$ the Polyakov action is still invariant under conformal transformations $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$. The world-sheet coordinates $\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$ and $\sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$ transform into

$$\begin{aligned} \tilde{\tau} &= \frac{1}{2}(\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-)), \\ \tilde{\sigma} &= \frac{1}{2}(\tilde{\sigma}^+(\sigma^+) - \tilde{\sigma}^-(\sigma^-). \end{aligned} \tag{3.51}$$

From here it follows that the coordinate $\tilde{\tau}$ can be an arbitrary solution of the wave equation

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\tilde{\tau} = 0.$$

This is exactly the equation of motion for the target space coordinates. Thus we can use the remaining gauge freedom to set $\tilde{\tau}$ equal to one of the coordinates X^μ. The coordinate $\tilde{\sigma}$ is then determined by eq.(3.51).

3.5.2 Light-cone Gauge

Let us first introduce the light-cone coordinates in spacetime as

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}),$$
$$X^I = X^i, \qquad i = 1, \ldots, D-2.$$

The light-cone gauge corresponds to the choice of the coordinate τ to be proportional to X^+, i.e.

$$X^+(\tau, \sigma) = x^+ + p^+\tau.$$

Then the $X^-(\tau, \sigma)$ coordinate is completely determined by the constraint equations $(\dot{X} \pm X')^2 = 0$ as

$$(\dot{X}^- \pm X^{-\prime}) = \frac{1}{2\alpha' p^+}(\dot{X}^I \pm X^{I\prime})^2. \tag{3.52}$$

One can introduce the transverse Virasoro modes for the coordinates X^I. In complete analogy with previous definitions (3.21) they are expressed as

$$L_n^{\perp} = \frac{1}{2}\sum_m \alpha^I_{n-m}\alpha^I_m, \qquad \bar{L}_n^{\perp} = \frac{1}{2}\sum_m \tilde{\alpha}^I_{n-m}\tilde{\alpha}^I_m,$$

where the repeated indices $I = 1, \ldots D-2$ denote summation over the transverse dimensions. Consequently, from the expansion of X^- coordinates

$$\dot{X}^- + X^{-\prime} = \sqrt{2\alpha'}\sum_n \tilde{\alpha}^-_n e^{-in\sigma^+},$$
$$\dot{X}^- - X^{-\prime} = \sqrt{2\alpha'}\sum_n \alpha^-_n e^{-in\sigma^-}$$

one can read off the equations for oscillator modes:

$$\sqrt{2\alpha'}\alpha^-_n = \frac{2}{p^+}L_n^{\perp}, \qquad \sqrt{2\alpha'}\tilde{\alpha}^-_n = \frac{2}{p^+}\bar{L}_n^{\perp}. \tag{3.53}$$

Hence, the only degrees of freedom left after imposing the light-cone gauge are: p^+, x_0^-, x_0^I, α_n^I. This choice of gauge is not covariant, however, and breaks Lorentz invariance, since the choice of components 0 and $D-1$ for defining the light-cone coordinates was completely arbitrary. One can show that in order for the quantum Lorentz generators to obey the Poincaré algebra, the constant values $a = 1$ and

$D = 26$ have to be chosen for bosonic strings.

3.6 Physical States

As was mentioned before, the Fock space built by creation operators α^μ_{-n}, $\mu = 0, \ldots, D$ acting on the vacuum state is not the physical state space which is spanned by all positive norm states $|\phi\rangle$ that satisfy the Virasoro condition $L_n |\phi\rangle = 0$, $n > 0$ and the mass-shell condition $(L_0 - a)|\phi\rangle = 0$. In this section we will show how to construct the physical state space without explicitly using the transverse oscillators α^I_{-n}.

3.6.1 Vertex Operators

The first step in order to unambigously define the physical state space is to associate an operator V_ϕ to every on-shell physical state $|\phi\rangle$. This will allow us to build new physical states from the old ones. What conditions should this new operator fulfill? First of all, it should be transformed into itself by the Virasoro algebra. Imposing this condition is necessary because the time evolution of any local quantum operator is determined by the corresponding Hamilton operator, which on the open string space is $L_0 - a$. Let us show that operators which satisfy this demand are actually the primary operators from conformal field theory[1]. Consider a field $V(\sigma = 0, \tau) \equiv V(\tau)$ on the open string Hilbert space. It is said that an operator has a conformal weight h if under an arbitrary change of variables $\tau \to \tau'$ it transforms like

$$V'(\tau') = \left(\frac{d\tau}{d\tau'}\right)^h V(\tau).$$

Written in infinitesimal form this transformation law becomes

$$\delta V(\tau) = -\varepsilon \frac{dV}{d\tau} + hV\frac{d\varepsilon}{d\tau}. \tag{3.54}$$

Rewriting eq. (3.26) at the point $\sigma = 0$ and substituting the Poisson bracket with commutator leads to

$$[L_m, X^\mu(\tau)] = -ie^{im\tau} X^\mu(\tau).$$

[1]We will discuss this in more detail in the next chapter. For now we will only use some of most essential properties of primaries.

Hence, the target space coordinates $X^\mu(\tau)$ have conformal weight $h = 0$, and we conclude that Virasoro operators generate transformations (3.54) with the infinitesimal parameter given by $\varepsilon = -ie^{im\tau}$. Thus for a field of arbitrary conformal weight (3.54) can be written

$$[L_m, V(\tau)] = e^{im\tau}\left(-i\frac{d}{d\tau} + mh\right)V(\tau). \tag{3.55}$$

If the operator $V(\tau)$ is expandable in a Fourier series, this condition can be imposed on the Fourier modes A_n as

$$[L_m, A_n] = (m(h-1) - n)A_{m+n}. \tag{3.56}$$

One can check that if $|\phi\rangle$ is a given physical state and the operator $V(\tau)$ has conformal dimension $h = 1$, then the state $|\phi'\rangle = A_0 |\phi\rangle$ is again a physical state. Therefore, we conclude that we are looking for an operator of conformal weight $h = 1$. In this case the transformation law (3.55) can be expressed as a total time derivative:

$$[L_m, V(\tau)] = -i\frac{d}{d\tau}(e^{im\tau}V(\tau)). \tag{3.57}$$

The second condition we have to impose on the vertex operator is that if at time τ and $\sigma = 0$ a physical state of momentum $-k^\mu$ is emitted by vertex operator $V(k, \tau)$, then it should increase the momentum of the initial state by an amount k^μ. This suggests that the vertex operator has to be proportional to $e^{ik\cdot x(\tau)}$ with $x(\tau)$ being the center of mass position of the string at time τ. So let us try the simplest expression we can come up with:

$$V(k, \tau) \equiv: e^{ik\cdot X(0,\tau)} : . \tag{3.58}$$

By straightforward calculation one can show that this operator has conformal weight $h = k^2/2$ for open strings. In the case $k^2 = 0$ this gives $h = 0$, and, hence, the expression (3.58) cannot be used as vertex operator describing the emission of a massless meson. However, one can show that the conjugated momenta $\dot{X}^\mu$ has $h = 1$, which is exactly what we are looking for. Thus, the next try should be the following

$$V_\zeta(k, \tau) = \zeta \cdot \frac{dX}{d\tau} \exp[ik \cdot X], \tag{3.59}$$

where $\zeta^\mu(k)$ is the polarization vector. If $k \cdot \zeta = 0$ then this expression has no short distance singularities in the operator product expansion and $V_\zeta(k,\tau)$ can be used as vertex operator. This condition on the polarization vector ensures that the vertex operators are in one-to-one correspondance with physical states.

3.6.2 Transverse Physical States

In this section we present the Del Giudice, Di Vecchia and Fubini (DDF) construction used to construct operators A_n^I, which when applied to the ground state give all possible transverse physical states. Note that we only refer to the states which correspond to the transverse oscillator modes α_n^I, where $I = 1, .., D-2$, as introduced in the light-cone quantization.

Let us first choose the ground state to be the tachyonic ground state $|p_0; 0\rangle$, which fulfills the mass shell condition $p_0^2 = 2$. Suppose that the tachyon is in a particular state with $p_0^+ = 1$, $p_0^- = -1$ and $p_0^I = 0$. Also define a vector k_0^μ with components $k_0^- = -1$, $k_0^+ = k_0^I = 0$, and, thus, $k_0 \cdot p_0 = 1$. This kinematic setup will be used throughout the construction of physical states.
We now define *allowed states* such that if the mass is given to be $\alpha' M^2 = N - 1$ then the momentum has to be $p^\mu = p_0^\mu - N k_0^\mu$. Any physical state obeying the mass shell condition can be Lorentz transformed into such a configuration.

As we discovered in the previous section, one can build new massless physical states from already existing physical state via applying the vertex operator (3.59). From our kinematical setup it follows that we are only studying states with a wave vector which is an integer multiple of the null vector defined above, i.e. $k^\mu = n k_0^\mu$. In this case the vertex operator for transverse polarizations is

$$V^I(nk_0, \tau) = \dot{X}^I(\tau) e^{inX^+(\tau)}, \tag{3.60}$$

where $X^+(\tau) = x^+ + \tau$. It follows that $V^I(nk_0, \tau) = V^I(nk_0, \tau + 2\pi)$. We define

$$A_n^I = \frac{1}{2\pi} \int_0^{2\pi} V^I(nk_0, \tau) d\tau = \frac{1}{2\pi} \int_0^{2\pi} \dot{X}^I(\tau) e^{inx^+} e^{in\tau}\, d\tau. \tag{3.61}$$

The operators A_n^I can be interpreted as the Fourier modes of a periodic operator which behaves as a primary field with weight $h = 1$ under the transformations generated by L_m at a given point $\sigma = 0$. Because of the periodicity condition, A_n^I

commutes with the Virasoro operators, as can be calculated from (3.57). As we will see later, this is actually the most important property an operator has to fulfill in order to generate physical states .

The following properties of operators A_n^I can be derived by direct computation:

$$\begin{aligned} &[L_m, A_n^I] = 0 \\ &[N, A_n^I] = nA_n^I \\ &[A_m^I, A_n^J] = m\delta_{IJ}\delta_{m+n} \\ &A_n^{I\dagger} = A_{-n}^I \\ &A_n^I \left|0; p_0\right\rangle = 0,\ n > 0. \end{aligned} \tag{3.62}$$

From here it is obvious that the A_n^I have the same properties as the transverse oscillators α_n^I and therefore states of the form

$$\left|f\right\rangle = A_{-1}^{I_1} A_{-2}^{I_2} ... A_{-m}^{I_m} \left|0; p_0\right\rangle \tag{3.63}$$

satisfy the Virasoro conditions and have $N = \sum rI_r$. In other words, these states are physical, linearly independent, and have positive metric. We will call generic states of the form (3.63) *DDF states*, and the space spanned by them we denote F. As the operators A_n^I are in one-to-one correspondence with the algebra of transverse oscillators, we can then conclude that they form a $D-2$ dimensional physical subspace of the complete Fock space.

3.6.3 No Ghost Theorem for $D = 26$ and $a = 1$

The purpose of this section is to show that there are no ghosts if we choose the spacetime dimension to be $D = 26$ and normal ordering constant $a = 1$. The idea of the proof is to show that all states in the complete Fock space built from oscillator modes as shown in (3.41) can be identified with DDF states, which are physical, positive norm states, according to the previous section. We will sketch the proof as it is necessary for further discussion, but only its main steps and results. Detailed proof can be found in [17].

Let us define the operators

$$K_m = k_0 \cdot \alpha_m, \tag{3.64}$$

where the scalar product is taken over all spacetime dimensions. This operator has the following properties

$$[K_m, L_n] = mK_{m+n}, \qquad [K_m, K_n] = 0 \tag{3.65}$$
$$K_n \left|f\right\rangle = 0, \qquad n > 0.$$

Here and henceforth we will take $\left|f\right\rangle$ to be a DDF state. We also define K to be a space spanned by all the states of the form

$$\left|k\right\rangle = \prod_{n=1}^{\infty} K_{-n}^{\mu_n} \left|f\right\rangle . \tag{3.66}$$

We now are going to explore the properties of the states built by acting on DDF states with operators L_{-n} and K_{-n}. We introduce

$$\left|\{\lambda, \mu\}, f\right\rangle \equiv L_{-1}^{\lambda_1} L_{-2}^{\lambda_2} \ldots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \ldots K_{-m}^{\mu_m} \left|f\right\rangle \tag{3.67}$$

with the eigenvalue P of the number operator defined as

$$P \equiv \sum r\lambda_r + \sum s\mu_s. \tag{3.68}$$

The ordering in (3.67) was chosen arbitrarily and this is possible due to the commutation relations of the L's and K's. Once an ordering is chosen we will stay to this convention throughout the calculation. Also note that the subscript m is the same for both the L's and K's. This is done only for the elegance of notation and denotes the highest order of operators K or L used to build a given state. It is still allowed that $\lambda_m = 0$ if $\mu_m \neq 0$.

We now claim that the states (3.67) are linearly independent. To show this consider the matrix of inner products of states (3.67) for a given value of P and some DDF state $\left|f\right\rangle$:

$$\begin{aligned} \mathcal{M}^P_{\{\lambda,\mu\};\{\lambda',\mu'\}} = \left\langle f\right| K_n^{\mu_n} \ldots K_1^{\mu_1} L_n^{\lambda_n} \ldots L_1^{\lambda_1} \\ L_{-1}^{\lambda'_1} \ldots L_{-m}^{\lambda'_m} K_{-1}^{\mu'_1} \ldots K_{-m}^{\mu'_m} \left|f\right\rangle , \end{aligned} \tag{3.69}$$

where $P = \sum r\lambda_r + \sum s\mu_s = \sum r\lambda'_r + \sum s\mu'_s$. One can then show that there exists an ordering of the states like $i = \{\lambda, \mu\} < j = \{\lambda', \mu'\}$ such that the matrix $\mathcal{M}^P_{ij}$

takes the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{pmatrix} . \tag{3.70}$$

The states (3.67) are then linearly independent because $\det(\mathcal{M}^P) \neq 0$.

There are two kinds of states with nonzero inner products: the states made out of L's or K's only, or states with an equal number of L's and K's. The latter restriction is needed because the evaluation process of the elements of $\mathcal{M}^P$ consists of commuting the operators past each other in order to get an L_0 or K_0 acting on the DDF state. If there would be more operators K than L, then an operator K_n coming from the left with $n > 0$ would hit $|f\rangle$ and give zero. After these remarks it is easy to define an appropriate ordering and to prove the claim.

The important remark has to be made that the presence of K operators is crucial. They are the key ingredients which ensure the non-singularity of matrix $\mathcal{M}^P$. The calculation of the determinant of inner product matrix for the states made out of L's only (Kac determinant) is of great importance in CFT and gives restrictions on the allowed values of the conformal weights h and the central charges c.

One can also check that any two states built from two orthogonal DDF states $|f\rangle$ and $|g\rangle$ which are L_0 eigenstates are also orthogonal, as are the states built upon them. This allows us to conclude that the states (3.67) made from all possible DDF states and $\{\lambda, \mu\}$ running over all strings of L's and K's are linearly independent.

Let us summarize. We have so far two sets of states. The first set is the Fock space built by acting on the string vacua with oscillators. A generic state in this space can be written as

$$\prod_{\rho=0}^{25} \prod_{n=1}^{\infty} (\alpha_{-n}^{\rho})^{\epsilon_{n,\rho}} |0\rangle . \tag{3.71}$$

The second set of states is the one introduced in (3.67). More explicitly, any such state can be written as a product

$$\prod_{n=1}^{\infty} L_{-n}^{\lambda_n} \cdot K_{-n}^{\mu_n} \cdot \prod_{I=1}^{24} \left(A_{-n}^{I}\right)^{\beta_{n,I}} |0\rangle . \tag{3.72}$$

We claim now that every state in the bosonic open string Fock space (3.71) can be expressed as a linear combination of basis states (3.72). To prove this one has to show that the number of states with a given eigenvalue $\langle N \rangle$ of the number operator

$$N = \sum_{\rho=0}^{25} \sum_{n=1}^{\infty} \alpha^{\rho}_{-n} \alpha_{n\rho} \tag{3.73}$$

is the same for both (3.71) and (3.72). For the Fock space states this gives

$$\langle N \rangle = \sum_{n,\rho} n \epsilon_{n,\rho} \tag{3.74}$$

and for the states (3.72)

$$\langle N \rangle = \sum_{n=1}^{\infty} n \left(\lambda_n + \mu_n + \sum_{I=1}^{24} \beta_{n,I} \right) . \tag{3.75}$$

The combinatorics of 26 ϵ's and one λ, one μ, and 24 β's is the same. Thus we can use the states (3.72) as the basis of the Fock space instead of (3.71). These states are not all physical though. What we have shown so far is only that they span the whole Fock space built from oscillators.

Further, let us define a *spurious state*. A state $|\psi\rangle$ is called spurios if it satisfies the constraint $(L_0 - a)\,|\psi\rangle = 0$ and is orthogonal to every physical state $|\phi\rangle$, i.e. $\langle \phi | \psi \rangle = 0$. We denote such states with $|s\rangle$, and call the space they span S. Every state of the form (3.72) is spurious if it has at least one operator L_n in it. The rest of the states belong to K since they contain the operators K only. Hence, any state $|\phi\rangle$ in the Fock space can be written as a sum

$$|\phi\rangle = |s\rangle + |k\rangle \, . \tag{3.76}$$

From here it follows that if $|\phi\rangle$ is an eigenstate of L_0, then $|s\rangle$ and $|k\rangle$ are also eigenstates of L_0 with the same eigenvalue. One can further show that if $|\phi\rangle$ is a physical state then $|s\rangle$ and $|k\rangle$ are also physical states. This is true, however, only if $D = 26$ because this value is used explicitly in the proof of the claim. The last

step is to consider a general form of a state (3.66). This can be written as a sum

$$|k\rangle = |f\rangle + \sum_{\alpha} \prod_{n=1}^{\infty} K_{-n}^{\mu_{n,\alpha}} |f_\alpha\rangle \equiv |f\rangle + |\tilde{k}\rangle, \tag{3.77}$$

and one can show that if $|k\rangle$ is physical, then the decomposition (3.77) simply becomes $|k\rangle = |f\rangle$. Hence, every physical state in the K space is a DDF state. Thus we conclude that every physical state $|\phi\rangle$ can be decomposed into a sum of a physical spurious state $|s\rangle$ and a DDF state $|f\rangle$ which by its construction is physical

$$|\phi\rangle = |s\rangle + |f\rangle\,. \tag{3.78}$$

Finally, we are able to prove that there are no ghosts in the physical Hilbert space which is a subspace of states spanned either by (3.71) or (3.72) fulfilling the Virasoro condition $(L_m - a\delta_{m0})\,|\phi\rangle = 0$, $\forall m \geq 0$:

$$\begin{aligned} \langle\phi|\phi\rangle &= (\langle s| + \langle f|)\,(|s\rangle + |f\rangle) \\ &= \langle s|s\rangle + \langle f|s\rangle + \langle s|f\rangle + \langle f|f\rangle \\ &= \langle f|f\rangle \geq 0. \end{aligned}$$

4

On Representations of the Virasoro Algebra in Conformal Field Theory

Here the construction of the highest-weight representation of the Virasoro algebra in conformal field theory is investigated. We first review the basic notions and techniques used in conformal field theory [19; 20]. Then we conclude the chapter with the calculation of the level density of highly excited states and compare the results with string theory. Finally we discuss the relation between this and black hole physics.

4.1 Classical Conformal Field Theory

4.1.1 Conformal Symmetry

By definition the conformal transformations are a subgroup of the diffeomorphism transformations $x^\mu \to x'^\mu$, under which the metric remains invariant up to an overall scale factor, i.e.

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \Omega(x) g_{\mu\nu}(x). \tag{4.1}$$

4. ON REPRESENTATIONS OF THE VIRASORO ALGEBRA IN CONFORMAL FIELD THEORY

An infinitesimal coordinate transformation can be written in terms of conformal Killing vector fields X_a^μ as

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) = x^\mu + \epsilon_a X_a^\mu.$$

For translations, rotations and dilations this yields[1]

$$\begin{aligned}
\xi^\mu_{(T)}(x) &= \epsilon^\nu X^\mu_\nu = \epsilon^\nu \delta^\mu_\nu \\
\xi^\mu_{(R)}(x) &= \epsilon_{(\nu\rho)} X^{\mu(\nu\rho)} = \epsilon_{(\nu\rho)}(\delta^{\mu\nu} x^\rho - \delta^{\mu\rho} x^\nu) \\
\xi^\mu_{(D)}(x) &= \epsilon x^\mu.
\end{aligned} \tag{4.2}$$

In d-dimensional ($d > 2$) space of signature (p, q) the conformal group is finite and the conformal algebra is isomorphic to $so(p+1, q+1)$.

The defining equation (4.1) for infinitesimal conformal coordinate transformations $x^\mu \to x^\mu + \varepsilon^\mu(x)$ in two dimensional Euclidean space reduces to the Cauchy-Riemann differential equations

$$\partial_1 \varepsilon_1 = \partial_2 \varepsilon_2, \qquad \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1. \tag{4.3}$$

If we complexify the Euclidean coordinates and coordinate transformations as

$$\begin{aligned}
z = x^1 + ix^2, \qquad & \bar{z} = x^1 - ix^2, \\
\varepsilon^z(z,\bar{z}) = \varepsilon(z,\bar{z})^1 + i\varepsilon^2(z,\bar{z}), \qquad & \bar{\varepsilon}^{\bar{z}}(z,\bar{z}) = \varepsilon^1(z,\bar{z}) - i\varepsilon^2(z,\bar{z}),
\end{aligned}$$

then the equations (4.3) imply holomorphic dependence of the conformal transformations $\varepsilon^z = \varepsilon^z(z)$ and $\bar{\varepsilon}^{\bar{z}} = \bar{\varepsilon}^{\bar{z}}(\bar{z})$. Therefore the two dimensional conformal transformations can be identified with analytic coordinate transformations

$$z \to f(z), \qquad \bar{z} \to \bar{f}(\bar{z}).$$

This allows us to treat z and $\bar{z}$ as two independent variables. By independent we mean that a priori $\bar{z} \neq z^*$. the condition $\bar{z} = z^*$ is only a section in our $\mathbb{C}^2$ space which recovers the initial 2d Euclidean space. Hence, we have complexified the

[1]The round brackets here are only indicating the distinction between different kinds of indices and have nothing to do with the symmetrization.

initial real 2d Euclidean space to a 2d complex space $\mathbb{R}^2 \to \mathbb{C}^2$. In two spacetime dimensions, then, the local[1] conformal group is the set of all analytic maps of the complex plane onto itself, which is obviously an infinite dimensional group.

4.1.2 Conformal Ward Identities

In a quantum field theory the main objects of interest are the correlation functions which are defined via the path integral as

$$\langle \phi(x_1)...\phi(x_n)\rangle = \frac{1}{Z}\int [d\phi]\, \phi(x_1)...\phi(x_n)e^{-S[\phi]}. \tag{4.4}$$

It is natural to demand the invariance of correlation functions under symmetry transformations that leave the action itself invariant. The consequences of some symmetry on correlation functions is expressed in the form of *Ward identities*. More precisely, Noether theorem states that the variation of the action $\delta S[\phi, \delta\phi] \equiv S[\phi + \delta\phi] - S[\phi]$ can be expressed in terms of a current j_a^μ as

$$\delta S = \int d^d x (\partial_\mu j_a^\mu)\epsilon_a. \tag{4.5}$$

This current is then conserved on the mass shell, i.e. where the classical equations of motion are satisfied,

$$\partial_\mu j^\mu = 0. \tag{4.6}$$

For correlation functions, conservation of the current leads to the Ward identities

$$\frac{\partial}{\partial x^\mu}\langle j_a^\mu(x)\phi(x_1)...\phi(x_n)\rangle = -i\sum_{j=1}^{n}\delta(x-x_i)\langle\phi(x_1)...G_a\phi(x_j)...\phi(x_n)\rangle\,, \tag{4.7}$$

where G_a is the generator of the symmetry transformation

$$\delta\phi = \phi' - \phi = -iG_a\epsilon_a.$$

This tells us that the current is conserved away from the insertions of the field ϕ.

Any field theory consistent with general relativity has to be diffeomorphism invariant. Therefore, let us consider an infinitesimal general coordinate transfor-

[1] Globally only Killing vectors corresponding to translations, rotations, dilations, and special conformal transformations are well-defined.

mation under which the matter fields transform like $\phi_i(x) \to \tilde{\phi}_i(x) = \phi_i(x) + \delta\phi_i(x)$. The variation of the action functional of matter, $S^m[g^{\mu\nu}, \phi]$, has to vanish

$$\delta S^m = \int d^d x \frac{\delta S^m}{\delta g^{\mu\nu}(x)} \delta g^{\mu\nu}(x) + \int d^d x \frac{\delta S^m}{\delta \phi_i(x)} \delta\phi_i(x) = 0. \tag{4.8}$$

We then introduce the energy-momentum tensor in a spacetime with Euclidean signature as

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S^m}{\delta g^{\mu\nu}}. \tag{4.9}$$

Then the Ward identities for diffeomorphism transformations in two dimensions reads

$$\sum_{j=1}^{n} \langle \phi(x_1)...\delta\phi(x_j)...\phi(x_n)\rangle = \frac{1}{2} \int d^2x \sqrt{g}\, \delta g_{\mu\nu}(x) \langle T^{\mu\nu}(x)\phi(x_1)...\phi(x_n)\rangle\,. \tag{4.10}$$

If we assume that the unperturbed background metric is flat $\delta_{\mu\nu}$, then the square root $\sqrt{g} = 1$, and the metric perturbations become $\delta g_{\mu\nu} = -(\xi_{\mu,\nu} + \xi_{\nu,\mu})$. Thus,

$$\begin{aligned} \sum_{j=1}^{n} \langle \phi(x_1)...\delta\phi(x_j)...\phi(x_n)\rangle &= -\int d^2x\, \partial_\mu \xi_\nu(x) \langle T^{\mu\nu}(x)\phi(x_1)...\phi(x_n)\rangle \\ &= -\int d^2x \left\{(\partial_\mu \epsilon_a) X_{a\nu} + \epsilon_a \partial_\mu X_{a\nu}\right\}(x) \cdot \\ &\qquad \cdot \langle T^{\mu\nu}(x)\phi(x_1)...\phi(x_n)\rangle\,, \end{aligned} \tag{4.11}$$

where in the second line the metric perturbations were expressed in terms of Killing vector fields $\partial_\mu \xi_\nu = \partial_\mu(\epsilon_a X_a^\mu)$. For conformal transformations given in equations (4.2) the second term vanishes only if the energy-momentum tensor is symmetric and traceless. Hence, this is a sufficient, but not a necessary, condition for a conformal symmetry to be generated by the energy-momentum tensor. As there are no counterexamples found so far, we will assume that classically the energy-momentum tensor is symmetric and traceless. So finally, after combining (4.7) and (4.11), we have

$$\int d^2x (\partial_\mu \epsilon_a) X_{a\nu}(x) \langle T^{\mu\nu}(x)\phi(x_1)...\phi(x_n)\rangle = \int d^2x \langle (\partial_\mu \epsilon_a) j_a^\mu \phi(x_1)...\phi(x_n)\rangle. \tag{4.12}$$

This gives the relation between the energy-momentum tensor and the conserved

current of the conformal symmetry of the action functional

$$j_{\mu a} = X_a^{\nu} T_{\mu\nu}. \tag{4.13}$$

In flat space, the energy-momentum tensor is the conserved current due to the translational symmetry with infinitesimal, position dependent parameter. Therefore, $\partial_\mu T^{\mu\nu} = 0$.

4.1.3 Generators of Conformal Transformations

In terms of the variables z and $\bar{z}$ the scale invariance and conservation of the energy-momentum tensor can be rewritten as[1]

$$\begin{gathered} \bar{\partial} T_{zz} + \partial T_{\bar{z}z} = 0, \qquad \partial T_{\bar{z}\bar{z}} + \bar{\partial} T_{z\bar{z}} = 0 \\ T_{\bar{z}z} = T_{z\bar{z}} = 0. \end{gathered} \tag{4.14}$$

Therefore, only the holomorphic and anti-holomorphic parts of the energy-momentum tensor $T(z) \equiv T_{zz}$ and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$ are non-zero. The Ward identity (4.10), rewritten with z and $\bar{z}$, takes the form

$$\delta_{\varepsilon\bar{\varepsilon}} \langle A \rangle = \frac{1}{2\pi i} \oint_C dz \varepsilon(z) \langle T(z) A[\phi] \rangle - \frac{1}{2\pi i} \oint_C d\bar{z} \bar{\varepsilon}(\bar{z}) \left\langle \bar{T}(\bar{z}) A[\phi] \right\rangle . \tag{4.15}$$

The integration contour encircles the origin and all the points inside the set of the fields $A[\phi] \equiv \phi(x_1)...\phi(x_n)$, as the identity (4.15) is identically zero elsewhere.

From complex analysis we know that every meromorphic ($\partial_{\bar{z}} \phi(z) = 0$) function can be expanded in a Laurent series. Hence, the coordinate change $\varepsilon(z)$ becomes

$$\varepsilon(z) = \sum_{n \in \mathbb{Z}} z^{n+1} \epsilon_n.$$

The generators of conformal transformations can then be defined as

$$\begin{aligned} L_n &= \frac{1}{2\pi i} \oint dz \, z^{n+1} T(z) \\ T(z) &= \sum_{n \in \mathbb{Z}} z^{-n-2} L_n. \end{aligned} \tag{4.16}$$

[1] the notation $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$ is introduced here

4.1.4 Primary Fields

The fields of greatest importance in conformal field theory are the ones transforming under conformal transformations in a way that the form $\phi_{h,\bar{h}}(z,\bar{z})dz^h d\bar{z}^{\bar{h}}$ is conformally invariant. Such a field $\phi_{h,\bar{h}}(z,\bar{z})$ is called a *primary field*. It follows that a primary field transforms as

$$\tilde{\phi}(\tilde{z},\tilde{\bar{z}}) = \phi(z,\bar{z})\left(\frac{dz}{d\tilde{z}}\right)^h\left(\frac{d\bar{z}}{d\tilde{\bar{z}}}\right)^{\bar{h}}$$

and $(h,\bar{h})$ is called a *conformal weight*. The infinitesimal form of this transformation law reads

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = [(h\partial\varepsilon + \varepsilon\partial) + (\bar{h}\bar{\partial}\bar{\varepsilon} + \bar{\epsilon}\bar{\partial})]\phi(\varepsilon,\bar{\varepsilon}). \tag{4.17}$$

The appropriate Laurent expansion of a primary field reads

$$\begin{aligned}\phi(z) &= \sum_{n\in\mathbb{Z}} z^{-n-h}\phi_n,\\ \phi_n &= \frac{1}{2\pi i}\oint dz\, z^{n+h-1}\phi(z).\end{aligned} \tag{4.18}$$

4.2 Radial Quantization of Conformal Field Theories

In order to cannonically quantize a conformal field, the notion of a time axis has to be introduced. So far we have been working with Euclidean coordinates described by a single complex coordinate $z = x^1 + ix^2$. We may proceed by parametrizing the complex z plane as follows:

$$z = e^w, \qquad w = \tau + i\sigma. \tag{4.19}$$

By restricting the range of σ coordinate to $\sigma \in [0, 2\pi)$, this corresponds to a mapping of a cylinder w to a complex plane z. The coordinate $\tau \in (-\infty, \infty)$ can then be regarded as time. The infinite past and future, $\tau = \pm\infty$, is mapped to the points $z = 0, \infty$ respectively. The circles of fixed radius around the origin on the z plane are interpreted as *equal time* slices. Hence, the time ordering required

in quantum mechanics translates into a radial ordering in conformal field theory. Therefore, we introduce the *radial ordering operator* $\mathcal{R}$

$$\mathcal{R}(A(z)B(w)) = \begin{cases} A(z)B(w), & \text{if } |z| > |w| \\ B(w)A(z), & \text{if } |z| < |w|. \end{cases} \tag{4.20}$$

The equal-time commutator is then defined as

$$[T_\varepsilon, \phi(w,\bar{w})] = \lim_{|z|\to|w|} \oint \frac{dz}{2\pi i} \varepsilon(z) T(z) \phi(w,\bar{w})$$

with the integration contour encircling the point w.

The conformal Ward identity (4.15) can be rewritten as

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\left(\oint [dz\, T(z)\varepsilon(z), \phi(w,\bar{w})] + \left[d\bar{z}\, \bar{T}(\bar{z})\bar{\varepsilon}(\bar{z}), \phi(w,\bar{w})\right]\right). \tag{4.21}$$

In analogy to the classical definition of conserved charge, let us define the charge corresponding to a conformal coordinate transformation $z \to z + \varepsilon(z)$ as

$$Q_\epsilon = \oint \frac{dz}{2\pi i} \varepsilon(z) T(z). \tag{4.22}$$

From the mode expansions (4.16) it follows that

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n \tag{4.23}$$

After rewriting the holomorphic part of the Ward identity (4.21) in terms of L_n

$$\delta_\epsilon \phi(z) = [\sum_{n\in\mathbb{Z}} \epsilon_n L_n, \phi(z)] \tag{4.24}$$

one recognizes the mode expansion coefficients L_n as generators of a local conformal transformation. This is in agreement with our knowledge from string theory, as there the Virasoro operators generated the remaining gauge transformations which were left after the conformal gauge was fixed. These symmetries corresponded to the conformal transformations of the worldsheet light-cone coordinates $\tau \pm \sigma$. Hence, string theory in conformal gauge is a conformal field theory.

4.2.1 Operator product expansion

We note that the conformal Ward identity (4.21) is only non-zero if there is a singularity in the operator product $\lim_{z\to w} T(z)\phi(w)$. The correlation functions of fields typically also have singularities at the points where the fields inside the correlator coincide. This corresponds to infinite quantum fluctuations once the field is localized. In general we call an expression, which reflects the singular behavior of the product of two or more local operators an *operator product expansion* (OPE).

Thus, for example, the OPE of a primary field with the energy-momentum tensor can be derived from (4.21) and is

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}). \tag{4.25}$$

"$\sim$" means that expressions like (4.25) are valid when the product is inserted into a correlation function. The OPE's for products of the stress tensor with primary fields and for products of the primary fields themselves contain all the information about dynamics and can actually substitute for a conformal field theory Lagrangian.

4.3 Central Charge and the Virasoro Algebra

The OPE for the energy-momentum tensor with itself is:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{4.26}$$

The factor 2 in front of the second term on the right hand side suggests that T is a quasi-primary field with conformal weight $h = 2$. This is violated by the anomalous term that has appeared due to the scale invariance condition of two point correlation functions. In other words, the first term gives the 2-point correlation function of the stress tensor

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4}.$$

The constant c is the central charge and depends on the conformal field theory for which the energy momentum tensor is computed for. Classically $c = 0$.

After combining equations (4.26) and (4.16) one can derive the following commutation relation defining the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{4.27}$$

with c being the central charge from the equation (4.26).

Let us consider the OPE (4.26) in its infinitesimal form

$$\begin{aligned}\delta_\epsilon T(w) = [T_\epsilon, T(w)] &= \oint_w dz\, \varepsilon(z)T(z)T(w) \\ &= [\varepsilon(w)\partial_w + 2\partial_w\varepsilon(w)]\, T(w) + \frac{c}{12}\partial_w^3\varepsilon(w).\end{aligned} \tag{4.28}$$

For a finite transformation $z = f(w)$ this gives

$$T(z) \to \tilde{T}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}S[z;w], \tag{4.29}$$

where we have introduced the *Schwartzian derivative*

$$S[f;w] = \frac{\partial_w f \partial_w^3 f - \frac{3}{2}(\partial_w^2 f)^2}{(\partial_w f)^2}. \tag{4.30}$$

For the conformal transformation $z = e^w$ from the cylinder (with coordinates w) to the plane (with coordinates z) one obtains the following relation between the corresponding energy-momentum tensors

$$T_{cyl}(z) = z^2 T(z) - \frac{c}{24} = \sum_{n\in\mathbb{Z}} L_n z^{-n} - \frac{c}{24}, \tag{4.31}$$

and, hence, the zero modes of the Virasoro algebra are shifted

$$(L_0)_{cyl} = (L_0)_{plane} - \frac{c}{24}. \tag{4.32}$$

Note that this also changes the Hamilton operator on the cylinder.

4.4 Hilbert Space of Conformal Fields

4.4.1 Operator-state Correspondence

The vacuum of a conformal field theory is defined as a state $|0\rangle$ which is invariant under global conformal transformations. This means that it is annihilated by operators L_{-1}, L_0, L_1. The condition that $T(z)\,|0\rangle$ is regular at $z = 0$ gives further restrictions

$$L_n\,|0\rangle = 0, \qquad n \geq -1.$$

This implies the vanishing of the vacuum expectation value of the energy-momentum tensor, i.e. $\langle 0|T(z)|0\rangle = 0$.

The CFT in-states are defined by applying the CFT operators to the vacuum,

$$|A_{in}\rangle = \lim_{z\to 0} A(z)\,|0\rangle \equiv A(0)\,|0\rangle\,.$$

The limit $z \to 0$ in radial quantization corresponds to $\tau \to \infty$. Therefore, we have assumed that the CFT fields are asymptotically free and can be used as CFT in-states.

The primary states are defined as $|h\rangle \equiv \phi_h(0)\,|0\rangle$, and are often called *highest-weight states*. From the OPE between the stress tensor and primary fields ϕ_h it follows that $L_0\,|h\rangle = h\,|h\rangle$, and likewise, we have $L_n\,|h\rangle = 0$, $\forall n > 0$. The excited states above the asymptotic state $|h\rangle$ can be obtained by applying the raising operators L_{-n}.

4.4.2 Highest-weight Representations. Verma Module

The simplest conformal field theories are characterized by a Hilbert space made of a finite number of representations of the Virasoro algebra. These include discrete statistical models at their critical points (Ising model, etc.). Such theories are called minimal models, and their correlation functions are completely determined once the corresponding partition function is known. It turns out that in order for a representation of the Virasoro algebra to be unitary, very severe restrictions are put on the central charge and highest-weight of the theory under consideration. Therefore, if the central charge is known, there is only a limited number of allowed values for the highest-weight. Hence, the partition function of a physical theory is

completely determined by algebraic methods. Therefore, the analysis of highest-weight representations is a very powerful tool.

Let us construct representations of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}. \tag{4.33}$$

We denote by $|h\rangle$ the highest-weight state, an eigenstate of operator L_0 with eigenvalue h,

$$L_0 |h\rangle = h |h\rangle . \tag{4.34}$$

From the commutation relations it follows that

$$L_0 (L_n |h\rangle) = (h-n) (L_n |h\rangle) ,$$

and, hence, the state $L_n |h\rangle$ is an eigenstate of L_0 operator. Since to every highest-weight state $|h\rangle$ there is related some primary field of conformal weight h, the operators L_n lower and raise (depending on the sign of n) the conformal dimension of the field. Therefore, the operators L_n with $n > 0$ are called *lowering* operators, and L_{-n} are called *raising* operators. The condition that the vacuum expectation value of the energy-momentum tensor has to vanish translates into the condition

$$L_n |h\rangle = 0, \qquad \forall\, n > 0.$$

It is important to note that in order to satisfy this condition it is sufficient to impose

$$L_1 |h\rangle = L_2 |h\rangle = 0, \tag{4.35}$$

since all higher level operators can be turned into linear combinations of L_1 and L_2 by repeated use of commutation relations. All other basis states of the representation can be obtained by successive application of raising operators on the highest-weight state:

$$L_{-n_1} L_{-n_2} \dots L_{-n_m} |h\rangle , \qquad 1 \le n_1 \le n_2 \le \dots \le n_m. \tag{4.36}$$

This state is then an eigenstate of operator L_0 with an eigenvalue

$$h' = h + \sum_{i=1}^{m} n_i = h + N.$$

We call such a state a *descendant state* of the highest-weight state $|h\rangle$ of level N. The Hermitian conjugate is defined as: $L_n^\dagger = L_{-n}$. The only non-zero inner products are among the descendant states of the same level.

The Verma module, generated by the set $\{L_n\}$ is denoted as $V(c,h)$ and is completely determined once the values of the conformal dimension and central charge are given.

Note the analogy with string theory. The mass shell condition $(L_0 - a)\,|\phi\rangle = 0$, together with the physicality condition $L_n\,|\phi\rangle = 0$, $\forall n > 0$ translates into saying that $|\phi\rangle$ is a highest-weight state of conformal weight a. It follows that the physical states of string theory are the highest-weight states of conformal field theory. We also note, that from the point of view of operator L_0 there is no difference between the states built as descendants by repeated application of the oscillator modes of the string α_{-n}^μ or the Virasoro operators L_{-n}. The L_0 eigenvalue is the same in both cases.

4.4.3 Singular Vectors and Spurious States

For certain values of the central charge c and the conformal weight h it might happen that the Verma module $V(c,h)$ is reducible. This is to say, that there is a subspace that is itself a representation of the Virasoro algebra. This is the case if among the descendants of $|h\rangle$ there exists a state $|\chi\rangle$ such that it is a highest-weight state itself, i.e. $L_n\,|\chi\rangle = 0$, $\forall n > 0$.

Such states are also called *null* states. They generate their own Verma module, which is a submodule of the initial module $V(c,h)$. The null states (and also their descendants) are orthogonal to any other state in the original Verma module, including themselves. To see this consider a descendant state of singular vector $|\chi\rangle$

$$L_{-r_1} L_{-r_2} \dots L_{-r_m}\,|\chi\rangle\,. \tag{4.37}$$

If $|\chi\rangle$ is itself a descendant of the original highest-weight state $|h\rangle$ of level N, then the level of the descendant (4.37) is $\sum_i r_i + N$. The only possible non-zero inner

product can come from the states of the same level, i.e.

$$\langle h| L_{k_n} \dots L_{k_1} L_{-r_1} \dots L_{-r_m} |\chi\rangle$$

with $\sum_i k_i > \sum_i r_i$. Therefore, one can bring the L_k's to the right and they will annihilate the highest-weight state $|\chi\rangle$.
The physical meaning of such subspaces is that the states of this submodule transform among themselves under conformal transformations. Hence, by identifying the states which differ from each other only by a state of zero norm, one is identifying two states which differ from each other only by conformal transformation. After having quotiened out all the zero norm states, one obtains an irreducible representation of the Virasoro algebra.

Again there is an analogy with string theory. In order to prove that the DDF states span the whole physical positive norm space we considered the orthogonal complement of the DDF state space F. This complement was built by acting on DDF states $|f\rangle$ with operators K_{-n} and L_{-n}. From the point of view of L_0 operators, they both were acting as raising operators. By this we mean that the level of the DDF states was raised by applying operators K_{-n} or L_{-n}. The main idea of the proof was to show that every physical, non-negative norm state in string Fock space can be written as

$$|\phi\rangle = |f\rangle + |s\rangle\,, \tag{4.38}$$

where $|s\rangle$ was proven to be a spurious state of zero norm. Then we concluded, that the only physical states with positive norm are the DDF states $|f\rangle$. However, only in the case of $D = 26$ it is possible to have enough null states for the splitting of every physical state of the open string Hilbert space as in eq. (4.38). Thus, the reason for the quite unnatural choice of 26 dimensions in the string theory can be derived from the purely algebraic considerations of reducible Verma modules.

4.5 Degeneracy of Highly Excited States

There are two common methods for determining the number of states at the energy level N for a specific conformal field theory. The first one is to use the partition function to obtain the level density. The second approach is to simply count the

number of different ways how to reach the level N starting from the vacuum state. We will, therefore, begin this section with considering the partition function on the torus. We will then recall how the modular invariance of the partition function can be used to derive the Cardy formula for the level density $\rho(N)$ of a given highest-weight representation [21].

4.5.1 Partition Function on the Torus

We have seen so far that the holomorphic and antiholomorphic sectors of a conformal field theory which is deefined on the whole complex plane completely decouple. Hence, each sector could in principle describe a distinct theory, which seems to be quite unphysical. As an example we recall that in string theory, although the left- and right-moving modes of the closed string were decoupled, still the level matching condition $L_0 = \bar{L}_0$ had to be imposed. The left-right sectors of a conformal field theory can be coupled through the geometry of the space. We will, therefore, study conformal field theories on the torus, which is topologically equivalent to the complex plane with periodic boundary conditions in two directions. Defining a lattice on a complex plane corresponds to providing two lattice vectors and identifying all the points which differ from each other by an integer linear combination of these vectors. Hence, to uniquely specify a lattice on the complex plane one only has to fix two complex numbers w_1 and w_2. As the partition function can be interpreted as zero point function, it has to be conformally invariant. Therefore, only the relative angle and ratio of the two complex numbers w_1 and w_2 are important. By setting $w_1 = 1$ one can define the modular parameter of the torus as $\tau = w_2 = \tau_1 + i\tau_2$ with $\tau_1 = \mathrm{Re}w_2$ and $\tau_2 = \mathrm{Im}w_2$. Recall that with the exponential mapping (4.19) the complex plane was mapped to an infinite cylinder. Hence, the first of the two identifications is already fixed, and the first of the lattice vectors w_1 points along the circumference of the cylinder. The second identification can be performed as follows - first "go" further up along the cylinder axis by a stretch equal to $\mathrm{Im}w_2$ and then "make a twist along" the circumference of the cylinder by $\mathrm{Re}w_2$.

The states of a conformal field theory on the torus are propagated by Hamilto-

nian and momentum operators, which are defined as

$$H_{cyl} = (L_0)_{cyl} + (\bar{L}_0)_{cyl}$$
$$P_{cyl} = (L_0)_{cyl} - (\bar{L}_0)_{cyl}.$$

The translation in Eucledian time corresponds to going in the $\mathrm{Im}\tau = \mathrm{Im}w_2$ direction and is done by the Hamilton operator. The translation in space is equivalent to moving along the circumference of the cylinder by a stretch $\mathrm{Re}\tau = \mathrm{Re}w_2$, and is generated by the momentum operator. The partition function is then the trace over all possible states on the torus. This yields the following expression

$$\begin{aligned} Z(\tau,\bar{\tau}) &= Tr\, e^{-2\pi \mathrm{Im}\tau H} e^{2\pi i \mathrm{Re}\tau P} \\ &= Tr\, e^{2\pi i \tau (L_0)_{cyl}} e^{-2\pi i \bar{\tau} (\bar{L}_0)_{cyl}}. \end{aligned} \tag{4.39}$$

This partition function is invariant under modular transformations by construction. Global modular transformations are generated by two basis elements $S : \tau \to \tau + 1$ and $T : \tau \to -\frac{1}{\tau}$. The latter will be of great importance for us. Rewriting the equation (4.39) in terms of $q \equiv e^{2\pi i \tau}$, and substituting (4.32) gives the final expression for the partition function on the torus

$$Z(\tau,\bar{\tau}) = Tr\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}. \tag{4.40}$$

4.5.2 Derivation of the Cardy Formula

The Cardy formula for the density of the eigenstates of the operator L_0 on the N-th level is

$$\rho(N) \approx \exp\left\{2\pi\sqrt{\frac{cN}{6}}\right\}, \tag{4.41}$$

where c is the central charge. The anti-holomorphic part is suppressed for notational simplicity. We will recall the key ideas of the proof of this formula. The full derivation can be found in [22].

First, we consider a partition function on the two-torus of modulus τ of the kind

$$Z'(\tau,\bar{\tau}) = Tr e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0} = \sum \rho(N,\bar{N}) e^{2\pi i N \tau} e^{-2\pi i \bar{N} \tau}. \tag{4.42}$$

As we have seen in previous section, this is not a modular invariant partition function. However, the weight $\rho(N, \bar{N})$ corresponds to the density of states with eigenvalues N, $\bar{N}$ for the operators L_0, $\bar{L}_0$ respectively. By contour integration one can extract the degeneracy

$$\rho(N, \bar{N}) = \frac{1}{(2\pi i)^2} \int \frac{dq}{q^{N+1}} \frac{d\bar{q}}{\bar{q}^{\bar{N}+1}} Z'(q, \bar{q}),$$

where we have substituted the parameter $q \equiv e^{2\pi i \tau}$. Let us further consider the holomorphic dependence only. We observe that the modular invariant partition function (4.40) is related to (4.42) by

$$Z'(\tau) = e^{\frac{2\pi i c}{24}\tau} Z(\tau).$$

In particular, the partition function $Z(\tau)$ is invariant under $\tau \to -1/\tau$. For $Z'(\tau)$ this translates into the following equality

$$Z'(\tau) = e^{\frac{2\pi i c}{24}\tau} Z(\tau) = e^{\frac{2\pi i c}{24}\tau} Z(-\frac{1}{\tau}) = e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} Z'(-\frac{1}{\tau}).$$

The level density is then

$$\rho(N) = \int d\tau \, e^{-2\pi i N \tau} e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} Z'(-1/\tau). \tag{4.43}$$

For large N, the extremum of the exponent is at $\tau \approx i\sqrt{\frac{c}{24N}}$. Evaluating the above integral with the saddle point approximation then gives the Cardy formula (4.41). However, this approach does not explicitly display which states are being counted.

4.5.3 Combinatorial Approach to the Counting of States

A more straightforward method to determine the density of states $\rho(N)$ is based on the combinatorics of creation operators. Consider as an example a single bosonic field, whose creation and annihilation operators α_n obey the algebra

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}.$$

The oscillator vacuum $|0\rangle$ is defined as

$$\alpha_n |0\rangle = 0, \qquad \text{for} \quad n > 0. \tag{4.44}$$

The operators α_{-n} are then used to create the excited states. Since $[L_0, \alpha_{-n}] = n\alpha_{-n}$, and $L_0 |0\rangle = 0$, it follows that

$$L_0 \prod_{j=1}^{m} \alpha_{-n_j} |0\rangle = \sum_{j=1}^{m} n_j \prod_{j=1}^{m} \alpha_{-n_j} |0\rangle .$$

The number of the eigenstates of L_0 at the excited level N can be expressed as the *number of partitions* $p(N)$ of an integer N into a sum of integers. For large integers N, the asymptotic behavior of the number of partitions is given by the formula [23]:

$$p(N) \sim \frac{1}{\sqrt{48}N} e^{2\pi\sqrt{\frac{N}{6}}}. \tag{4.45}$$

This agrees with the Cardy formula for $c = 1$.

4.5.4 Level Density of Physical States in String Theory

Every physical state in the open string state space can be written as

$$\prod_{I=1}^{24} \prod_{n=1}^{\infty} (\alpha_{-n}^I)^{\lambda_n^I} |0\rangle , \tag{4.46}$$

where $|0\rangle$ denotes the oscillator vacuum (4.44). We neglect the momentum of the center of mass of the string as this is of no relevance to this discussion. The λ_n^I's are the occupation numbers, and the open string number operator $\hat{N} = \sum_m \alpha_{n-m}^I \alpha_m^I$ acting on the state (4.46) returns the eigenvalue $N = \sum_{n,I} n\lambda_n^I$. We note that this is simply an extension of the previous discussion of one scalar field to $D - 2 = 24$ scalar fields. The generalization of (4.45) to a partition with $D - 2$ "colors" gives

$$p_{D-2}(N) \sim \frac{1}{\sqrt{2}} \left(\frac{D-2}{24}\right)^{\frac{(D-2)+1}{4}} N^{-\frac{(D-2)+3}{4}} \exp\left(2\pi\sqrt{\frac{(D-2)N}{6}}\right).$$

For the case $D = 26$ this yields

$$p_{24}(N) \sim \frac{1}{\sqrt{2}} N^{-27/4} \exp\left(2\pi\sqrt{\frac{24N}{6}}\right). \tag{4.47}$$

Hence, the bosonic string theory can effectively be treated as a conformal field theory of central charge $c = D - 2$.

4.5.5 Applications to $2+1$ dimensional black holes

It has been shown that the asymptotic symmetry group of 2+1 dimensional gravity with a negative cosmological constant $\Lambda = -1/l^2$ is generated by two copies of the Virasoro algebra [24], with central charges

$$c_L = c_R = \frac{3l}{2G}.$$

These central charges are classical and appear in the Hamilton formalism as the canonical generators of the asymptotic symmetries. Hence, the degrees of freedom of a black hole horizon in 2+1 dimensions at the spatial infinity are described by a conformal field theory. The asymptotic growth of the level density then yields:

$$S = 2\pi\sqrt{\frac{cN_R}{6}} + 2\pi\sqrt{\frac{cN_L}{6}}. \tag{4.48}$$

For a three dimensional black hole of Bañados, Teitelboim and Zanelli (BTZ) of mass M and angular momentum J this relation can be rewritten as

$$S = \pi\sqrt{\frac{l(lM+J)}{2G}} + \pi\sqrt{\frac{l(lM-J)}{2G}}, \tag{4.49}$$

where the relations $M = \frac{1}{l}(L_0 + \bar{L}_0)$ and $J = L_0 - \bar{L}_0$ have been used [25]. This is in exact agreement with the Bekenstein-Hawking entropy for a BTZ black hole

$$S = \frac{\pi}{G}\sqrt{Gl(Ml + \sqrt{M^2l^2 - J^2})}.$$

However, in order for this scenario to be acceptable one must first show that the BTZ black hole can be obtained as a solution of a consistent theory of quantum gravity on AdS_3 (see [26] and references therein).

5

Quantum Black Holes

The state of current knowledge about the quantization of black holes will be presented in this chapter. This discussion follows closely a paper of Bekenstein [6]. The basic heuristic properties of the algebra of black hole horizon area will be presented.

5.1 The Area Spectrum

The goal of this section is to give some justification of why the area of a black hole should be quantized. We also wish to discuss some main characteristics of the eigenvalue spectrum of the horizon area.

Let us recall the notion of an adiabatic invariant in classical mechanics. It is a quantity $A(p,q)$ which changes little during a time period while the Hamiltonian H changes significantly. Ehrenfest has shown that all action integrals of the form $A = \oint pdq$ are adiabatic invariants. In the old Bohr-Sommerfeld theory Jacobi actions are quantized in integers $\oint pdq = 2\pi n\hbar$. Combining this knowledge suggests that any classical adiabatic invariant corresponds to a quantum operator with a discrete spectrum. We wish to argue that the horizon area of a black hole is an adiabatic invariant.

Let us consider a classical Kerr-Newman black hole of mass M, electric charge Q, angular momentum J and area A satisfying the relation

$$M^2 = \frac{A}{16\pi}\left(1 + \frac{4\pi Q^2}{A}\right)^2 + \frac{4\pi J^2}{A}. \tag{5.1}$$

Now we will imagine a point particle approaching the black hole. On the black hole horizon the particle has a turning point. This process is happening very slowly and thus can be considered a classically reversible process [27]. The change in the area of the black hole ΔA can be calculated by varying the expression (5.1). The result shows that the area remains unchanged. Hence, it is indeed a classical adiabatic invariant. This in correspondence with the theorem of Ehrenfest suggests that the horizon area of a quantum black hole must have a discrete eigenvalue spectrum.

In quantum mechanics the notion of a point particle does not exist anymore and one has to take into account the finite size of the particle. The smallest radius b that one can associate to an elementary particle with mass m is of the order of its Compton length $b = \xi\hbar/m$. Here ξ is a number of order unity. Bekenstein showed that the absorption of a particle necessarily involves an increase in the horizon area [6]. The reason for this is that the center of mass of the particle cannot be localized directly on the black hole horizon, and is instead a distance b away from it. The minimal change in area is then

$$(\Delta A)_{min} = 8\pi\mu b = 8\pi\xi\hbar \equiv \alpha l_{Pl}^2. \tag{5.2}$$

Observe that the minimal increase in the horizon area is universal and does not depend on any other properties of the black hole. Therefore $(\Delta A)_{min}$ can be regarded as the spacing between the eigenvalues of area operator A. Thus, the spectrum of area eigenvalues is positive and uniformly spaced

$$a_n = \alpha l_P^2(n + \eta); \qquad \eta > -1,\ n \in \mathbb{N}. \tag{5.3}$$

The parameter η has been introduced in order to take into account the possible vacuum area.

5.2 The Origins of Black Hole Entropy

The results of the previous section suggest that a black hole of given area A consists of $n = A/\alpha l_P^2$ equal pieces. If one assumes that all these area patches are equivalent then there is an equal number k of microscopic eigenstates, hidden to an external observer inside of every area patch. Hence, the total number of different quantum

states of the horizon is

$$N = k^{A/\alpha l_P^2}. \tag{5.4}$$

We can imagine then, that the black hole is built by adding one piece of area at every moment of time. Each such area quantum is an independent degree of freedom by itself. Therefore, one can interpret the black hole entropy statistically as the logarithm of the number of different quantum states the black hole is made of. One can consider the black hole horizon area as being split into small parts, each of them containing k "particles". The entropy is reflected in our missing knowledge about the microscopic internal state of the black hole. This leads to the expression

$$S = \frac{\ln k}{\alpha} \frac{A}{l_P^2} \tag{5.5}$$

for the entropy, which when compared to the Bekenstein-Hawking entropy

$$S_{BH} = \frac{1}{4} A + \text{ const} \tag{5.6}$$

suggests that $\alpha = 4 \ln k$.

If one now accepts proportionality between black hole entropy and horizon area, then one can follow Mukhanovs approach to determine the degeneracy g_n of a given area eigenvalue a_n [8]. From statistical physics we know that the entropy of some macroscopic configuration is given by the logarithm of the number of its microstates. As the black hole entropy is directly related to the horizon area, one can calculate the degeneracy of a particular area eigenvalue as

$$\begin{aligned} g_n &= \exp(S_{BH}) \\ &= \exp(a_n / 4\alpha l_P^2) \\ &= g_1 \exp(\frac{\alpha}{4}(n-1)), \end{aligned} \tag{5.7}$$

where $g_1 \equiv \exp(\frac{\alpha}{4}(\eta + 1))$ denotes the degeneracy of black hole ground state. In order for g_n to be an integer number, some restrictions on g_1 and α have to be imposed. In the original paper of Mukhanov a nondegenerate black hole ground state was assumed, i.e. $g_1 = 1$, which lead to degeneracy $g_n = 2^{n-1}$. However, a doubly degenerate black hole ground state would be a better choice, as this allows us to set the constant η to zero. Thus the choice of $g_1 = 2$ corresponds to setting

$\eta = 0$, $\alpha = 4 \ln 2$. This gives the following area spectrum and its degeneracy:

$$a_n = 4 l_p^2 \ln 2 \cdot n, \qquad n \in \mathbb{N}; \tag{5.8}$$

$$g_n = 2^n. \tag{5.9}$$

A simple explanation for this degeneracy comes from combinatorics. It describes in how many ways one can get up the staircase to the n-th level. In the case of horizon area the "staircase" would be the levels of area eigenvalues. Finally, we rewrite the expression of the degeneracy as

$$g_n = e^{n \ln 2}. \tag{5.10}$$

5.3 Predictions Due to the Existence of Discrete Area Spectrum

In order to see what macroscopic consequences the quantization of the black hole horizon area has, let us consider a black hole with zero charge and zero angular momentum. Then we have from (5.1) the following relation:

$$M^2 = \frac{A}{16\pi} \qquad \Rightarrow \qquad M \sim \sqrt{n}. \tag{5.11}$$

This implies that due to the discreteness of the area spectrum, the mass spectrum is also discrete. Taking into accout the formula (5.8) one can derive the following mass level spacing

$$\omega_0 \equiv \Delta M/\hbar = \frac{\ln 2}{8\pi M}. \tag{5.12}$$

By analogy with atomic physics a black hole should be able to make a spontaneous transition from mass level n to $n-1$. This could explain the Hawking radiation in the limit of highly excited black holes. However, from eq. (5.12) we are expecting a line spectrum with frequencies, which are multiples of ω_0. This is in contradiction with the continuous thermal spectrum of the Hawking radiation. In the case of very massive black holes the intensity of radiation is exponentially surpressed, but for primordial black holes the first lines should be detectable. There have been some attempts to coincide the line spectrum with the continuous Hawking spectrum

by arguing, for example, that the spectral lines are broadened so much that the spectrum becomes continuous. However, the broadening of a line is negligible [9], i.e.

$$\frac{\Delta\omega}{\omega_0} \sim 0.019\gamma, \tag{5.13}$$

where γ is a numerical factor of order unity. This suggests that the line spectrum is in fact sharp.

5.4 An Algebraic Description of Black Holes

Let us assume that a quantum black hole state is desribed by the eigenvalues of the set of operators $\{Q, \boldsymbol{J}^2, J_z, A\}$. The spectrum of the first three operators is well known from atomic physics. The eigenvalues are, Q $\{qe;\, q = \text{integer}\}$, $\boldsymbol{J}^2$ $\{j(j+1)\hbar^2\}$, and J_z $\{m\hbar = -j\hbar, -(j-1)\hbar, \ldots, (j-1)\hbar, j\hbar\}$, with j being a nonnegative integer or half-integer. We have no information whatsoever about the spectrum of the area operator. It is, however, possibile to derive its characteristic properties from algebraic considerations. Bekenstein proposed the following axioms:

Axiom 1: The horizon area operator A is positive, semi-definite, and has a discrete spectrum $\{a_n;\, n \in \mathbb{N}\}$. The degeneracy of an eigenvalue g_n is independent of the quantum numbers j, m, q.

The fact that the degeneracy is independent of quantum numbers coming from other operators is due to the consideration that the operators A, Q, $\boldsymbol{J}^2$, and J_z mutually commute.

Axiom 2: There exist some operators $R_{\lambda s}$ with $\lambda = \{\text{njmq}\}$ which play the role of creation operators of the black holes, i.e. a state $R_{\lambda s}\,|\text{vac}\rangle$ is a black hole with horizon area a_n, corresponding eigenvalues of angular momentum, and some internal quantum nuber s.

This internal quantum number distinguishes between states with equal horizon area. We do not know any characteristics of this number, but we do know that the area spectrum is degenerate in the sense that for an external observer the internal configurations of a black hole remain invisible.

Axiom 3: The operators A, $\boldsymbol{J}$, Q, $R_{\lambda s}$ and $[A, R_{\lambda s}]$ form a closed, linear, infinite-dimensional nonabelian algebra.

The properties of operator $R_{\lambda s}$, listed in Axiom 2, impose severe restrictions on the commutation relations between $R_{\lambda s}$ and the rest of operators. After demanding that the vacuum state is invariant under rotations and has zero area and charge, one obtains the following commutation relations

$$[J_z, R_{\lambda s}] = m_\lambda \hbar R_{\lambda s}, \tag{5.14}$$

$$[J_\pm, R_{\lambda s}] = \sqrt{j_\lambda(j_\lambda+1) - m_\lambda(m_\lambda \pm 1)}\hbar R_{njm_\lambda \pm 1 qs} \tag{5.15}$$

$$[Q, R_{\lambda s}] = q_\lambda e R_{\lambda s}. \tag{5.16}$$

After considering the Jacobi identities involving A, $R_{\lambda s}$, and any of the remaining operators it follows that the commutation relations of operator $[A, R_{\lambda s}]$ are exactly the same as the ones for $R_{\lambda s}$, listed above. This allows us to write the operator as

$$[A, R_{\lambda s}] = a_\lambda R_{\lambda s} + T_{\lambda s} \tag{5.17}$$

with $T_{\lambda s}\left|\text{vac}\right\rangle = 0$ in order for $[A, R_{\lambda s}]\left|\text{vac}\right\rangle = a_\lambda R_{\lambda s}\left|\text{vac}\right\rangle$ to be fulfilled. One can further show that $[A, T_{\lambda s}]$ must be expressible as a linear combination of other operators A, $\boldsymbol{J}$, Q, $R_{\lambda s}$, $T_{\lambda s}$. Hence, it is possible to redefine the operator $R_{\lambda s}$ in such a way that the new operator $R^{\text{new}}_{\lambda s}$ creates the same black hole states as $R_{\lambda s}$ and satisfies the commutation relation

$$[A, R^{\text{new}}_{\lambda s}] = a_\lambda R^{\text{new}}_{\lambda s}. \tag{5.18}$$

Henceforth we use only the "new" creation operator and drop the superscript.

5.5 Properties of the Area Operator

One can check that

$$AR_{\kappa s}R_{\lambda t}\left|\text{vac}\right\rangle = (a_\kappa + a_\lambda)R_{\kappa s}R_{\lambda t}\left|\text{vac}\right\rangle, \tag{5.19}$$

i.e. the horizon area of the state $R_{\kappa s}R_{\lambda t}\left|\text{vac}\right\rangle$ is the sum of the horizon areas of the states $R_{\kappa s}\left|\text{vac}\right\rangle$ and $R_{\lambda t}\left|\text{vac}\right\rangle$. This suggests that the eigenvalues of the area operator are additive. Nonetheless, it could still be possible that the state $R_{\kappa s}R_{\lambda t}\left|\text{vac}\right\rangle$ describes two black holes and not one. In this case the relation (5.19)

would be a triviality. It can, however, be shown that, when one operates on a vacuum state with the commutator $[R_{\kappa s}, R_{\lambda t}]$, the resulting state describes a *one*-black hole state:

$$[R_{\kappa s}, R_{\lambda t}] \left|\text{vac}\right\rangle = \left|\text{one BH}\right\rangle . \tag{5.20}$$

It follows that

$$R_{\kappa s} R_{\lambda t} \left|\text{vac}\right\rangle = \left|\text{one BH}\right\rangle + \left|\text{two BH}\right\rangle . \tag{5.21}$$

Hence, the eigenvalue $a_\kappa + a_\lambda$ can be used to describe a one-black hole as well as a two-black hole state.

By taking hermitian conjugate of equation (5.18) one obtains

$$[A, R^\dagger_{\lambda s}] = -a_\lambda R^\dagger_{\lambda s} \tag{5.22}$$

and thus

$$A R^\dagger_{\kappa s} R_{\lambda t} \left|\text{vac}\right\rangle = (a_\lambda - a_\kappa) R^\dagger_{\kappa s} R_{\lambda t} \left|\text{vac}\right\rangle . \tag{5.23}$$

Several conclusions can be drawn here. First, the operators $R^\dagger_{\kappa s}$ annihilate the vacuum and, thus, can be considered as lowering operators. This further implies that the state $R^\dagger_{\kappa s} R_{\lambda t} \left|\text{vac}\right\rangle$ is a purely one-black hole state, as a lowering operator cannot have created an extra black hole. Second, positive differences of one-black hole area eigenvalues are also allowed eigenvalues of a black hole. This also implies that only integer area eigenvalues are allowed, and that the set $\{n a_1;\, n \in \mathbb{N}\}$ spans the entire spectrum of eigenvalues of A.

6

Diffeomorphism algebra in gravity

The goal of this chapter is to determine the quantum physical states of a two dimensional spacelike surface embedded in a four dimensional spacetime. When we quantize a scalar field the Fock space is built by creation and annihilation operators, which are determined by the mode expansion of the solution of wave equation satisfied by a free scalar field. In general relativity the gravitational field equations are very nonlinear, so there exist no obvious analogue of creation and annihilation operators.

As we have seen in string theory, the target space coordinates satisfy the wave equations. However, the situation there is complicated by the existence of constraints due to the diffeomorphism invariance of the worldsheet hypersurface. Since, in general relativity the diffeomorphism constraints satisfy an algebra similar to the algebra of string theory constraints, one could try to use the methods developed in string theory to study the diffeomorphism invariant states in gravity.

6.1 Discretization of constraints

Let us parametrize the spacetime in such a way that the first two spacelike coordinates parametrize a two-dimensional surface, and thus the diffeomorphism transformations of this surface are generated by the first two constraints. The mutual Poisson bracket is given by

$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \mathcal{H}_j(x)\frac{\partial}{\partial x^i}\delta(x,y) - \mathcal{H}_i(y)\frac{\partial}{\partial y^j}\delta(x,y), \tag{6.1}$$

where i, j take values 1, 2 corresponding to the coordinates on the surface.

In quantum theory the constraints become operators and the Poisson brackets are replaced by commutators as $[\dots] = i\{\dots\}$. For simplicity let us consider a two dimensional surface with the topology of a torus $\mathbb{T}^2$. Then the constraints can be expanded in a Fourier series as:

$$\begin{aligned}\mathcal{H}_1(x) &= \sum_{n=-\infty}^{\infty} L^1_{nm} \mathrm{e}^{inx^1+imx^2}, \\ \mathcal{H}_2(x) &= \sum_{n=-\infty}^{\infty} L^2_{kl} \mathrm{e}^{ikx^1+ilx^2}.\end{aligned} \tag{6.2}$$

Taking into account that $\delta(x,y) = \sum_{n,m} e^{in(x^1-y^1)+im(x^2-y^2)}$, and substituting into (6.1), we find that the operators L^i_{nm} obey the following algebra:

$$\begin{aligned}[L^1_{nm}, L^1_{kl}] &= (n-k)L^1_{n+k,m+l}, \\ [L^2_{nm}, L^2_{kl}] &= (m-l)L^2_{n+k,m+l}, \\ [L^1_{nm}, L^2_{kl}] &= mL^1_{n+k,m+l} - kL^2_{n+k,m+l}.\end{aligned} \tag{6.3}$$

6.2 On Quantum Anomalies of 2D Diffeomorphism Algebra

Because of normal ordering ambiguities, there could be some kind of central extension of the diffeomorphism algebra. Here we will consider several possibilities. Let us first recall the central extension of Virasoro algebra in the one dimensional case.

6.2.1 Central Extension in One Dimension

Consider the 1D Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}. \tag{6.4}$$

By redefining the L_0 operator as $L'_0 := L_0 - c/24$, the term linear in m can be removed, as the commutator (6.4) becomes

$$[L_m, L_{-m}] = 2mL'_0 + \frac{c}{12}m^3. \tag{6.5}$$

From now on we will skip the linear term and use the relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}. \tag{6.6}$$

Setting $m = 0$ in the mode expansion of the $\mathcal{H}_1(x) = \mathcal{H}_1(x^1, x^2)$ constraint (6.2) corresponds to restricting to the one-dimensional hypersurface $x^2 = 0$:

$$\mathcal{H}_1(x) = \mathcal{H}_1(x^1, 0) = \sum_n L_{n0}\mathrm{e}^{inx^1}.$$

Hence, the operators L^1_{n0} have to obey the algebra (6.6) and satisfy

$$\left[L^1_{n0}, L^1_{k0}\right] = (n-k)L^1_{n+k,0} + \frac{c}{12}n^3\delta_{n+k,0}. \tag{6.7}$$

This imposes severe conditions on the possible central extensions of the two dimensional diffeomorphism algebra. In the following we denote $L^1_{nm} \equiv L_{nm}$ for brevity and use notation $L_{nm} = L_{nM}$ to distinguish between "active" and "passive" indices.

To prove the non-existence of central extension for the two dimensional Virasoro algebra we will use the Jacobi identity

$$[L_{nM}, [L_{kL}, L_{sT}]] + [L_{kL}, [L_{sT}, L_{nM}]] + [L_{sT}, [L_{nM}, L_{kL}]] = 0, \tag{6.8}$$

which has to be satisfied by the operators in order to form a closed algebra. The only possible central extension of the one dimensional algebra, which obeys (6.8) is of the form $n(n^2-1)$ as in (6.4).

6.2.2 Central Extension for the Two Dimensional Diffeomorphism Algebra

In this section we prove, that there is no central extension for two dimensional Virasoro algebra.

- Proposition 1

  The most general central extension, which would still satisfy (6.8), can be written as

$$[L_{nM}, L_{kL}] = (n-k)L_{n+k,M+L} + n^3\delta_{n+k,0}f(M,L) + g(M,L). \tag{6.9}$$

  Proof

  Assume that instead of functions $f(M,L)$, $g(M,L)$ we have $\tilde{f}(n,k|M,L)$ and $g = \tilde{g}(n,k|M,L)$ and demand that (6.9) reduces to (6.7) in the case $M = L = 0$. It follows that

$$\begin{aligned} &\tilde{f}(n,k|0,0) = a, \quad a \in \mathbb{R} \\ &\Rightarrow \quad \tilde{f}(n,k|M,L) = f(M,L) \qquad \text{with} \qquad f(0,0) = a. \end{aligned}$$

  Similarly, $\tilde{g}(n,k|0,0)$ has to satisfy

$$g_0(n,k) \equiv \tilde{g}(n,k|0,0) = bn(n^2-1), \qquad b \in \mathbb{R}.$$

  From this it follows then that the function $\tilde{g}$ can be separated into $\tilde{g}(n,k|M,L) = g_0(n,k) + g(M,L)$ with $g(0,0) = 0$. The function g_0 can be absorbed into the term $n^3\delta_{n+k,0}f(0,0)$ and L_0. Therefore we asssume that $g_0 \equiv 0$ and $f(0,0) = \frac{c}{12}$. Hence,

$$\tilde{g}(n,k|M,L) = g(M,L) \qquad \text{with} \qquad g(0,0) = 0.$$

- Proposition 2

$$f(M,L) = f(L,M), \tag{6.10}$$

$$g(M,L) = -g(L,M). \tag{6.11}$$

  Proof

  Use the antisymmetry of the commutator, i.e. $[L_{nM}, L_{kL}] = -[L_{kL}, L_{nM}]$.

This gives

$$\begin{aligned}
[L_{nM}, L_{kL}] &= (n-k)L_{n+k,M+L} + n^3\delta_{n+k,0}f(M,L) + g(M,L), \\
[L_{kL}, L_{nM}] &= (k-n)L_{n+k,M+L} + k^3\delta_{n+k,0}f(L,M) + g(L,M) \\
&= -(n-k)L_{n+k,M+L} - n^3\delta_{n+k,0}f(L,M) + g(L,M).
\end{aligned}$$

Comparing both expressions leads to $f(M,L) = f(L,M)$ and $g(M,L) = -g(L,M)$ q.e.d.

- Proposition 3

$$f(M,L) = \tilde{f}(M+L), \tag{6.12}$$
$$g(M,L) \equiv 0. \tag{6.13}$$

Proof
According to our assumption (6.9) and the following discussion, the functions $f(M,L)$ and $g(M,L)$ are defined to be mutually independent. Therefore, the Jacobi identity (6.8) also has to be satisfied independently. Hence, two equations have to be fulfilled:

$$\begin{aligned}
\big((k-s)n^3 f(M,L+T) + (s-n)k^3 f(L,T+M) + (n-k)s^3 f(T,M+L)\big) \cdot \\
\cdot \delta_{n+k+s,0} = 0, \\
(k-s)g(M,L+T) + (s-n)g(L,T+M) + (n-k)g(T,M+L) = 0.
\end{aligned}$$

As f and g do not dependen on "active" indices, the above equations can be satisfied only if

$$\begin{aligned}
f(M,L+T) = f(L,T+M) = f(T,M+L), \\
g(M,L+T) = g(L,T+M) = g(T,M+L).
\end{aligned}$$

By setting $T = 0$ these become

$$f(M,L) = f(L,M) = f(0,M+L) \equiv \tilde{f}(M+L), \tag{6.14}$$
$$g(M,L) = g(L,M) = g(0,M+L). \tag{6.15}$$

Equation (6.14) proves the first part of claim. Taking into account the antisymmetry of g, the second equation becomes $g(M,L) = -g(M,L)$. Hence, $g(M,L) \equiv 0$. Note, that from eq. (6.14) it follows that

$$f(M,-M) = f(0,0) = \frac{c}{12} \qquad \forall\, M.$$

Thus, the most general two dimensional Virasoro algebra with a central extension obeys the algebra

$$[L_{nM}, L_{kL}] = (n-k)L_{n+k,M+L} + n^3\delta_{n+k,0} f(M+L), \tag{6.16}$$

$$[M_{Nm}, M_{Kl}] = (m-l)M_{N+K,m+l} + nm^3\delta_{m+l,0}\bar{f}(N+K), \tag{6.17}$$

where $M_{Nm} \equiv L^2_{nm}$. The second relation was written by analogy, with active and passive indices exchanged. The functions f and $\bar{f}$ can be different.

- Proposition 4

  If

$$[L_{nm}, M_{kl}] = mL_{n+k,m+l} - kM_{n+k,m+l}, \tag{6.18}$$

  then $f = \bar{f} = 0$.

  Proof

  The Jacobi identities

$$[L_{nm},[M_{kl},M_{st}]] + [M_{kl},[M_{st},L_{nm}]] + [M_{st},[L_{nm},M_{kl}]] = 0,$$
$$[M_{mn},[L_{lk},L_{ts}]] + [L_{lk},[L_{ts},M_{mn}]] + [L_{ts},[M_{mn},L_{lk}]] = 0$$

  lead to the conditions

$$(sl^3 - kt^3)\bar{f}(n+k+s)\delta_{m+l+t,0} = 0,$$
$$(sl^3 - kt^3)f(n+k+s)\delta_{m+l+t,0} = 0.$$

  Since, both equations have to be satisfied for arbitrary values of indices, it follows that both functions have to be zero.

- Proposition 5

  Assume the following central extension of the commutator $[L, M]$:

$$[L_{Nm}, M_{kL}] = mL_{N+k,m+L} - kM_{N+k,m+L} + \tilde{g}(m, k|N, L), \qquad (6.19)$$
$$[M_{kL}, L_{Nm}] = kM_{k+N,L+m} - mL_{k+N,L+m} + \tilde{g}(k, m|L, N),$$

  where capital letters denote "passive" indices again. The arguments in $\tilde{g}$ are grouped according to their "active" or "passive" action.
  Then $\tilde{g}(m, k|N, L) = g(m + L|k, N)$ with $g(m + L|k, N) = -g(m + L|N, k)$.
  Proof
  Consider the Jacobi identity:

$$\begin{aligned} 0 &= [L_{nm}, [M_{kl}, M_{st}]] + [M_{kl}, [M_{st}, L_{nm}]] + [M_{st}, [L_{nm}, M_{kl}]] \\ &= (l - t)\tilde{g}(m, k + s|n, l + t) - m\tilde{g}(k, t + m|l, s + n) + mg(s, m + l|t, n + k) + \\ &\quad + (sl^3 - kt^3)\delta_{m+l+t,0}\bar{f}(k + s + n) + (L,\ M \text{ terms}). \end{aligned}$$

  By setting $l = t$ we obtain

$$0 = -m(\tilde{g}(k, m + l|l, s + n) - \tilde{g}(s, m + l|l, n + k)) + l^3(s - k)\delta_{m+2l,0}\bar{f}(k + s + n).$$

  For $m \neq -2l$ this gives the following equality

$$\tilde{g}(k, m + l|l, s + n) = \tilde{g}(s, m + l|l, n + k).$$

  It can be satisfied only if $\tilde{g}(m, k|n, l) = g(m + l|k, n)$.
  Hence, the commutators (6.19) become

$$[L_{Nm}, M_{kL}] = mL_{N+k,m+L} - kM_{N+k,m+L} + g(m + L|N, k), \qquad (6.20)$$
$$[M_{kL}, L_{Nm}] = kM_{k+N,L+m} - mL_{k+N,L+m} + g(L + m|k, N).$$

  From the antisymmetry $[L_{Nm}, M_{kL}] = -[M_{kL}, L_{Nm}]$ it follows that $g(m + L|k, N) = -g(m + L|N, k)$.

- Proposition 6

  $g(t|n, k) = 0, \qquad \forall t \neq 0 \quad \Rightarrow \quad g(t|n, k) = \delta_{t,0}\bar{g}(n + k) \quad \Rightarrow \quad \bar{g}(n) \equiv 0.$

Proof

Consider the Jacobi identity with two "passive" and one "active" index.

$$\begin{aligned} 0 &= [L_{n0}, [M_{l0}, M_{0t}]] + [M_{l0}, [M_{0t}, L_{n0}]] + [M_{0t}, [L_{n0}, M_{l0}]] \\ &= [L_{n0}, -tM_{lt}] + [M_{0t}, -lM_{l+n,0}] \\ &= -tg(t|n,l) - lt^3\delta_{t,0}\bar{f}(l+n) + (L,\ M \text{ terms}). \end{aligned}$$

The second term always vanishes, and, therefore, $g(t|n,l) = 0$, $\forall t \neq 0$. We can then write it in the form $g(t|n,k) = \delta_{t0}h(n,k)$. Returning back to the initial Jacobi identity

$$0 = [L_{nm}, [M_{kl}, M_{st}]] + [M_{kl}, [M_{st}, L_{nm}]] + [M_{st}, [L_{nm}, M_{kl}]]$$

it follows that

$$(l-t)h(n,k+s) + (l+t)h(k,s+n) - (l+t)h(s,n+k) + (sl^3 - kt^3)\bar{f}(k+n+s) = 0.$$

Hence, it can only be fulfilled if $h(n,k) = \bar{g}(n+k)$. After inserting this in the expression above, it follows that

$$(l-t)\bar{g}(n+k+s) + (sl^3 - kt^3)\bar{f}(n+k+s) = 0.$$

This cannot be satisfied. Therefore, we conclude, that $\bar{g}(n) \equiv 0$.

It follows from proposition 6.2.2 that $g = f = \bar{f} \equiv 0$. Hence, there is no central extension for the two dimensional Virasoro algebra.

6.3 Non-central Extensions of the 2D Virasoro Algebra

As it was found by Moody and Larsson [28; 29; 30], the algebra can be supplemented with a non-central extension. This means that instead of adding some function of c-numbers, as in the case of a central extension, one can extend the algebra (6.3) by adding certain operators. The commutators derived by Larsson, rewritten for

the two dimensional case, are

$$\begin{aligned}
\left[L^1_{nm}, L^1_{kl}\right] &= (n-k)L^1_{n+k,m+l} - nk(c_1+c_2)(nS^1_{n+k,m+l} + mS^2_{n+k,m+l}) \\
\left[L^2_{nm}, L^2_{kl}\right] &= (m-l)L^2_{n+k,m+l} - ml(c_1+c_2)(nS^1_{n+k,m+l} + mS^2_{n+k,m+l}) \\
\left[L^1_{nm}, L^2_{kl}\right] &= mL^1_{n+k,m+l} - kL^2_{n+k,m+l} - (c_1mk + c_2nl)(nS^1_{n+k,m+l} + mS^2_{n+k,m+l})
\end{aligned} \tag{6.21}$$

with non-central extensions defined via the operators S^μ. These obey the commutation relations:

$$\begin{aligned}
\left[L^1_{nm}, S^1_{kl}\right] &= lS^2_{n+k,m+l} \\
\left[L^1_{nm}, S^2_{kl}\right] &= -kS^2_{n+k,m+l} \\
\left[L^2_{nm}, S^1_{kl}\right] &= -lS^1_{n+k,m+l} \\
\left[L^2_{nm}, S^2_{kl}\right] &= kS^1_{n+k,m+l} \\
\left[S^i_{nm}, S^j_{kl}\right] &= 0.
\end{aligned} \tag{6.22}$$

and satisfy the identity

$$nS^1_{nm} + mS^2_{nm} = 0, \tag{6.23}$$

which follows from the antisymmetry of the commutator algebra, $[L_{nm}, L_{kl}] = -[L_{kl}, L_{nm}]$.

Some restrictions on the operators S^i and constants c_i follow from the 1D Virasoro algebra. Consider the commutator (6.21) with $m = l = 0$:

$$\left[L^1_{n0}, L^1_{k0}\right] = (n-k)L^1_{n+k,0} - n^2k(c_1+c_2)S^1_{n+k,0} \tag{6.24}$$

From the defining relation (6.23) it follows that

$$nS^1_{n0} = 0, \quad \forall n$$

which can be satisfied for all values of n only if S^1_{n0} is proportional to Kronecker delta. The same argument applies to S^2_{0m} and thus

$$\begin{aligned}
S^1_{n0} &= S^1_{00}\delta_{n0} \\
S^2_{0m} &= S^2_{00}\delta_{m0}.
\end{aligned} \tag{6.25}$$

Substituting in the commutators gives

$$\begin{aligned}\left[L^1_{n0}, L^1_{k0}\right] &= (n-k)L^1_{n+k,0} + n^3(c_1+c_2)S^1_{00}\delta_{n+k,0} \\ \left[L^2_{0m}, L^2_{0l}\right] &= (m-l)L^2_{0,m+l} + m^3(c_1+c_2)S^2_{00}\delta_{m+l,0}.\end{aligned} \tag{6.26}$$

Comparison of (6.7) and (6.26) yields the following relations for the constants in a non-central extension

$$\begin{aligned}(c_1+c_2)S^1_{00} &= \frac{c}{12} \\ (c_1+c_2)S^2_{00} &= \frac{\bar{c}}{12},\end{aligned} \tag{6.27}$$

where c and $\bar{c}$ are central charges for L^1 and L^2 respectively. It follows from here that the operators S^i_{00} are numbers and, hence, commute with L^i's.

6.4 Eigenspace of the Constraint Operators

In a conformal field theory of conformal weight h and central charge c all information necessary to find the correlation functions is encoded in its highest-weight representation space (Verma module $V(c,h)$), which is spanned by the eigenstates of L_0 operator. By following this analogy let us explore the properties of the space spanned by the eigenstates of L^i_{00}.

6.4.1 Eigenspace of Decoupled L^1_{0J} and L^2_{I0}

Let us consider the Virasoro algebra

$$\begin{aligned}[L^1_{nm}, L^1_{kl}] &= (n-k)L^1_{n+k,m+l}, \\ [L^2_{nm}, L^2_{kl}] &= (m-l)L^2_{n+k,m+l}, \\ [L^1_{nm}, L^2_{kl}] &= 0,\end{aligned} \tag{6.28}$$

where for simplicity, we have assumed that the constraint operators L^1_{nm} and L^2_{nm} commute, and that there are no extensions. Denote by $|J\rangle$ an eigenstate of L^1_{0J}

with eigenvalue λ_J and by $|I\rangle$ an eigenstate of L^2_{I0} with eigenvalue μ_I

$$L^1_{0J}\,|J\rangle = \lambda_J\,|J\rangle\,.$$
$$L^2_{I0}\,|I\rangle = \mu_I\,|I\rangle\,.$$

Consider the following commutator

$$\begin{aligned}\left[L^1_{0J}, L^1_{-nK}\right]|J\rangle &= nL^1_{-n,K+J}\,|J\rangle \\ &= L^1_{0J}L^1_{-nK}\,|J\rangle - L^1_{-nK}L^1_{0J}\,|J\rangle \\ &= L^1_{0J}L^1_{-nK}\,|J\rangle - \lambda_J L^1_{-nK}\,|J\rangle\,.\end{aligned}$$

For $J = 0$ it follows that

$$L^1_{00}L^1_{-nK}\,|\lambda\rangle = (\lambda + n)L^1_{-nK}\,|\lambda\rangle\,,$$

where $\lambda \equiv \lambda_0$ and $|\lambda\rangle \equiv |J = 0\rangle$. Hence, $L^1_{-nK}\,|\lambda\rangle$ is an eigenstate of L^1_{00} with eigenvalue $\lambda + n$. Note that the second index in operator L^1_{-nK} has no influence on the eigenvalues of L^1_{00}, and, as long as operators L^1, L^2 commute, it does not contribute to the eigenvalues of L^2_{I0} either. Therefore, following the notation in the previous section, we have denoted the second index in L^1_{nK} with a capital letter in order to indicate that it is "passive". Nevertheless, only in the case $J = 0$, are the states $L^1_{-nK}\,|J\rangle$ eigenstates of L^1_{0J}. The results for L^2_{I0} are analogous, and can be summarized as follows:

$$\begin{aligned}L^1_{00}L^1_{-nK}\,|\lambda\rangle &= (\lambda + n)L^1_{-nK}\,|\lambda\rangle \\ L^1_{00}L^1_{nK}\,|\lambda\rangle &= (\lambda - n)L^1_{nK}\,|\lambda\rangle \\ L^2_{00}L^2_{L,-m}\,|\mu\rangle &= (\mu + m)L^2_{L,-m}\,|\mu\rangle \\ L^2_{00}L^2_{Lm}\,|\mu\rangle &= (\mu - m)L^2_{Lm}\,|\mu\rangle\,.\end{aligned}$$

As expected, the eigenspaces of L^1_{00} and L^2_{00} are decoupled. For each fixed value K, for the set $\{L^1_{00} : L^1_{nK},\, \forall n \in \mathbb{Z}\}$, some sort of highest-weight representation can be defined with the highest-weight state $|\lambda\rangle$ satisfying:

$$L^1_{00}\,|\lambda\rangle = \lambda\,|\lambda\rangle\,.$$

The descendants are built by successively applying the raising operators L^1_{-nK}, $\forall n > 0$ on the highest-weight state. The gauge invariance condition, which has to be imposed is then $L^1_{nK}|\lambda\rangle = 0$, $\forall n > 0$. We note that

$$L^1_{00}\left(L^1_{0K}|\lambda\rangle\right) = \lambda\left(L^1_{0K}|\lambda\rangle\right).$$

Hence, the eigenvalue λ is double degenerate for each fixed K. However, the set $\{L^1_{00} : L^1_{nK}, \forall n \in \mathbb{Z}\}$ cannot be interpreted as one copy of the Virasoro algebra labeled by K. The reason for this is the fact that this set does not form a closed algebra:

$$[L^1_{nK}, L^1_{mK}] = (n-k)L^1_{n+m,2K}.$$

The Verma module of the whole Virasoro algebra of operators L^1, defined by the commutators (6.28), is spanned by the descendant states, which are obtained by repeated action on the highest-weight state $|\lambda\rangle$ with all possible raising operators L^1_{-nK}, $\forall K \in \mathbb{Z}$. However, the degeneracy of each eigenvalue of L^1_{00} is infinite. This is very unphysical and as such we will no longer discuss the decoupled two dimensional diffeomorphism algebra.

6.4.2 Eigenspace of Coupled L^1_{00}, L^2_{00} without Central Extension

Let us now take into account the non-commutativity of the constraint operators L^1 and L^2, and consider the algebra

$$\begin{aligned}
[L^1_{nm}, L^1_{kl}] &= (n-k)L^1_{n+k,m+l},\\
[L^2_{nm}, L^2_{kl}] &= (m-l)L^2_{n+k,m+l},\\
[L^1_{nm}, L^2_{kl}] &= kL^2_{n+k,m+l} - mL^1_{n+k,m+l}.
\end{aligned} \tag{6.29}$$

In this section we will investigate the highest-weight representation of the algebra (6.29) and what is the increase in the degeneracy $\Gamma(N)$ of level N.

We define the state $|\lambda\mu\rangle$ as

$$L^1_{00}|\lambda\mu\rangle = \lambda|\lambda\mu\rangle, \qquad L^2_{00}|\lambda\mu\rangle = \mu|\lambda\mu\rangle. \tag{6.30}$$

As before, one finds the following relations:

$$\begin{aligned} L^1_{00}L^i_{-nM}\left|\lambda\mu\right\rangle &= (\lambda+n)L^i_{-nM}\left|\lambda\mu\right\rangle, \\ L^1_{00}L^i_{nM}\left|\lambda\mu\right\rangle &= (\lambda-n)L^i_{nM}\left|\lambda\mu\right\rangle, \\ L^2_{00}L^i_{N,-m}\left|\lambda\mu\right\rangle &= (\mu+m)L^i_{N,-m}\left|\lambda\mu\right\rangle, \\ L^2_{00}L^i_{Nm}\left|\lambda\mu\right\rangle &= (\mu-m)L^i_{N,m}\left|\lambda\mu\right\rangle, \end{aligned} \tag{6.31}$$

with $i = 1, 2$. Hence, the operators L^1_{00} and L^2_{00} have the same eigenspace, as they commute. The corresponding eigenvalues are different though.
In order to construct the highest-weight representation space of the Virasoro algebra (6.29), one has to point out one generator A_0 which is diagonal in the representation space. This operator determines the highest-weight of the Verma module and defines the notion of raising and lowering operators of the eigenvalues of A_0. The choice of A_0 for the 1d Virasoro algebra was unambiguous, as there were no two generators which commute. Therefore, we have chosen the diagonal operator to be $A_0 = L_0$:

$$L_0\left|h\right\rangle = h\left|h\right\rangle, \qquad L_n\left|h\right\rangle = 0, \quad \forall n > 0.$$

The basis for the other states in the representation space was obtained by applying the raising operators L_n.

In string theory and conformal field theory one has two decoupled Virasoro algebras $\{L_n\}$ and $\{\bar{L}_n\}$. In order to extend the basis of the representation space to both eigenstates of L_0 and $\bar{L}_0$, the diagonal operator A_0 of the representation was defined as $A_0 = L_0 + \bar{L}_0$. This choice was justified because the operator $L_0 + \bar{L}_0$ corresponds to the closed string Hamiltonian. However, we could have, in principle, defined a different A'_0 of the form

$$\begin{aligned} A'_0 &= aL_0 + b\bar{L}_0, \quad a,\, b \in \mathbb{N}, \\ A'_0\left|h\bar{h}\right\rangle &= (ah + b\bar{h})\left|h\bar{h}\right\rangle. \end{aligned}$$

By this choice the highest-weight of the representation is changed, $h + \bar{h} \to ah + b\bar{h}$. The A'_0 eigenvalue of descendants of level $N_{tot} = N + \bar{N}$ is then

$$N' = a(h + N) + b(\bar{h} + \bar{N}).$$

Still, the different values of a, b do not change the degeneracy of the level N_{tot}. In practice, the choice $a = b$ corresponds to some overall factor. If $a \neq b$, we are introducing some kind of anisotropy. Hence, it seems that setting $a = b = 1$ is indeed the most natural choice.

Among the generators of the algebra (6.29), there are two mutually commuting operators L^1_{00} and L^2_{00} with the same eigenspace. In the analogy with conformal field theory we define the diagonal operator A_0 of the highest-weight representation as

$$A_0 = L^1_{00} + L^2_{00} \qquad \Rightarrow \qquad \left(L^1_{00} + L^2_{00}\right)|\lambda\mu\rangle = (\lambda + \mu)\,|\lambda\mu\rangle\,.$$

Hence, the state $|\lambda\mu\rangle$ is a highest weigth state with respect to operator A_0 of weight $\lambda + \mu$. A basis state of the representation space can be written in general form as

$$|\lambda'\mu'\rangle = \tilde{\prod_{i=0}^{l}}(L^1_{0,-i})^{\alpha_{0i}}(L^1_{-1,-i})^{\alpha_{1i}}\ldots(L^1_{-n_i,-i})^{\alpha_{n_i i}} \tilde{\prod_{j=0}^{m}}(L^2_{0,-j})^{\beta_{0j}}(L^2_{-1,-j})^{\beta_{1j}}\ldots(L^2_{-n_j,-j})^{\beta_{n_j j}}\,|\lambda\mu\rangle\,, \tag{6.32}$$

with $\alpha_{00} = \beta_{00} \equiv 0$. The tilde over the products means that the operators for different values of i are ordered in ascending order from left to right. This is possible due to the commutation relations and once the convention is chosen it has to remain fixed in order to ensure that the eigenstates (6.32) are linearly independent. Indices α_{ij} (β_{ij}) count the number of times the operator $L^1_{-i,-j}$, ($L^2_{-i,-j}$) was used.

Note, that only the operators $L^i_{-n,-m}$ with negative indices were used as the raising operators in order to insure, that the state $|\lambda\mu\rangle$ has the smallest A_0 eigenvalue among the basis states of the representation space. This condition can be written as

$$L^i_{nm}\,|\lambda\mu\rangle = 0, \quad \forall\, n, m > 0. \tag{6.33}$$

This is equivalent to demanding that the diffeomorphism constraints vanish in the operator sense, i.e. that for any two physical states $|\phi\rangle$, $|\psi\rangle$

$$\langle\psi|L^i_{nm}|\phi\rangle = 0, \qquad \forall\, n \in \mathbb{Z}.$$

The A_0 eigenvalue of the state (6.32) is then

$$N = \lambda + \mu + N_{L^1} + N_{L^2}, \tag{6.34}$$

$$N_{L^1} = \sum_{i=0}^{l} \sum_{k=0}^{n_i} (k+i)\alpha_{ki}, \tag{6.35}$$

$$N_{L^2} = \sum_{j=0}^{m} \sum_{k=0}^{n_j} (k+j)\beta_{kj}, \tag{6.36}$$

where N_{L^i} counts only the eigenvalues of A_0 which arise from the operators $L^i_{-n,-m}$ in the state (6.32).

6.4.3 Degeneracy

To find the degeneracy $\Gamma(N)$ of a given eigenvalue $N = N_{L^1} + N_{L^2}$ it is useful to first find the degeneracy of the eigenvalue N_{L^1}. The degeneracy $\Gamma(N)$ can then be found later as

$$\Gamma(N) = \sum_{N_{L^1}=0}^{N} \Gamma(N_{L^1})\Gamma(N - N_{L^1}).$$

In order to simplify the notation we will skip the superscript and denote the eigenvalue N_{L^1} simply as N_L.

Let us subdivide the degeneracy of the value N_L as

$$\Gamma(N_L) = \sum_{n=1}^{N_L} \Gamma_n(N_L),$$

where $\Gamma_n(N_L)$ denotes the degeneracy of the eigenvalue N_L which arises from a state which is built by n operators L_{-p-r}. An example for a state which contributes to $\Gamma_3(8)$ would be:

$$L_{-2-1}L_{-30}L_{-1-1}\left|0\right\rangle.$$

Hence, our task now is to calculate the degeneracy $\Gamma_n(N_L)$ of a given value of N_L when the number n of the operators L, which are used to build the states, is fixed. We proceed by counting the number of different ways one can distribute the value N_L among n operators L. This corresponds to the number of partitions of an integer number N_L into a sum of exactly n integers, and it is denoted by $p_n(N_L)$.

For the example considered above $p_3(8) = 5$, as it can be seen from:

$$\begin{aligned} 8 &= 1+1+6 \\ 8 &= 1+2+5 \\ 8 &= 1+3+4 \\ 8 &= 2+2+4 \\ 8 &= 2+3+3 \end{aligned} \tag{6.37}$$

The order of the different terms in the sum does not matter. Once the splitting is known, we have to consider each of them separately. Therefore we denote these by $\alpha^k(n, N_L)$, where the index $k = 1, \ldots, p_n(N_L)$. For (6.37) this means:

$$\begin{aligned} \alpha_1 &= \{1,1,6\} \\ \alpha_2 &= \{1,2,5\} \\ \alpha_3 &= \{1,3,4\} \\ \alpha_4 &= \{2,2,4\} \\ \alpha_5 &= \{2,3,3\} \end{aligned}$$

We will now consider one of the strings $\alpha_k(n, N_L)$. It denotes one of the possible ways of writing the number N_L as a sum of exactly n integers. α_k is only a label of one of such splittings. Hence, every α_k denotes a set $A_{\alpha_k}(n, N_L)$ of states which are built in a certain way. Every state of the form

$$L_{-p_1,-r_1} \ldots L_{-p_n,-r_n} |0\rangle$$

belongs to one of the sets A_{α_k}. For example, the state

$$L_{-1-2}L_{-30}L_{-1-1}|0\rangle \quad \in \quad A_{\alpha_5}(3,8). \tag{6.38}$$

We now denote the numbers in every string as $\alpha_k = \{\mu_1, \mu_2, \ldots, \mu_n\}$. Each number μ_j is a label associated to each operator L_{-p-r} and gives the value of the sum of the indices p and r, hence, $\mu_j = p + r$. For a given μ_j there is a certain amount of possibilities how it can be obtained. For example, if $\mu_j = 3$, it can originate from

the following operators:

$$\begin{aligned} \mu_j = 3: \qquad & L_{0-3} \\ & L_{-1-2} \\ & L_{-2-1} \\ & L_{-30} \end{aligned}$$

The order of the indices does matter and, thus, the degeneracy of μ_j is $\mu_j + 1$. Hence, effectively we can replace the "two-index-labeling" of an operator L_{-p-r} to "colored-index-labeling" $(L_{\mu_j})^a$, where a tells which one of the $\mu_j + 1$ possible configurations is used. Index a will be called the *color* of operator $(L_{\mu_j})^a$. For the above example we then have

$$\begin{aligned} \{0,-3\} &= 1, & L_{0-3} &= (L_3)^1 \\ \{-1,-2\} &= 2, & L_{-1-2} &= (L_3)^2 \\ \{-2,-1\} &= 3, & L_{-2-1} &= (L_3)^3 \\ \{-3,0\} &= 4, & L_{-3,0} &= (L_3)^4 \end{aligned}$$

Hence, every element μ_j in the string α_k can be picked in $\mu_j + 1$ different colors labeled by $a(\mu_j)$. Therefore a state which belongs to the set $A_{\alpha_k}(n, N_L)$ can be written as

$$(L_{\mu_1})^{a_{\mu_1}} (L_{\mu_2})^{a_{\mu_2}} \dots (L_{\mu_n})^{a_{\mu_n}} |0\rangle \,, \tag{6.39}$$

where $\sum_{j=1}^{n} \mu_j = N_L$ and $a_{\mu_j} = 1, \dots, \mu_j + 1$. The number $g(\alpha_k)$ of different states for a given string $\alpha_k(n, N_L)$ is then

$$g(\alpha_k) = \prod_{j=1}^{n} (\mu_j + 1).$$

However, if some of the μ_j's are equal, this diminishes the number of distinct states, which belong to the set A_{α_k}. This is because the colors of the equal μ_j's are the same and some of the states (6.39) will differ from each other only by the order in which the different L operators appear in the string. If we have $\mu_j = \mu_i = 3$ then,

for example, only one of the products

$$(L_3)^1(L_3)^2 = L_{0-3}L_{-1-2}$$
$$(L_3)^2(L_3)^1 = L_{-1-2}L_{0-3}$$

can be a part of a properly ordered basis state of the representation space of the 2d diffeomorphism algebra.

Suppose that the element μ_i appears in a given string α_k exactly l times. Then the number of distinct states in the set A_{α_k} is

$$g(\alpha_k) = \prod_{\substack{j=1 \\ \mu_j \neq \mu_i}}^{n} (\mu_j + 1) C^l(\mu_i + 1),$$

where

$$C^l(\mu_i + 1) = \frac{(\mu_i + l)!}{l!\mu_i!}$$

are the *combinations with repetitions*. They count the number of different ways how to choose l elements out of a set of $\mu_i + 1$ elements, if the order does not matter and repetitions are allowed.

Now, if there are several values of μ_j which appear more than once in the string α_k (e.g. $\{2, 2, 3, 3\}$), we have to take this into account as well. We denote by s the number of repeated elements, each of which appears l_i times and $i = 1, \ldots, s$. Then the number of distinct states in the set A_{α_k} is

$$g(\alpha_k) = \prod_{\substack{j=1 \\ \mu_j \neq \mu_i}}^{n} (\mu_j + 1) \cdot \prod_{i=1}^{s} C^{l_i}(\mu_i + 1). \tag{6.40}$$

The indices μ_j, l_i, and s are, however, dependent on the specific string α_k chosen, hence, we add an index k to each of these indices: μ_{jk}, l_{ik}, s_k. The total degeneracy of the eigenvalue N_L which is distributed among n operators L is then

$$\Gamma_n(N_L) = \sum_{k=1}^{p_n(N_L)} g(\alpha_k).$$

This corresponds to the number of *restricted, colored partitions* of an integer num-

ber N_L in exactly n parts with additional condition that the number of colors depends on the specific partition. The generating function for colored partitions in the non-restricted case is well-known [31]. However, there is no general solution of our knowledge for the case of interest.

The total degeneracy of the value N_L is

$$\Gamma(N_L) = \sum_{n=1}^{N_L} \sum_{k=1}^{p_n(N_L)} g(\alpha_k). \tag{6.41}$$

The non-trivial task is to calculate $g(\alpha_k)$, as we need to determine the values of μ_{jk}, l_{ik}, and s_k for every string α_k. Still, it is possible to calculate the degeneracy $\Gamma(N_L)$ nummerically by following the steps which were explained above. The results are presented in Figure 6.1. From there we conclude that for large values of N the asymptotic behavior of the degeneracy seems to be

$$\Gamma(N) \sim e^{2N^{3/4}}.$$

However, it seems that for higher values of N the degeneracy will deviate from these asymptotics.

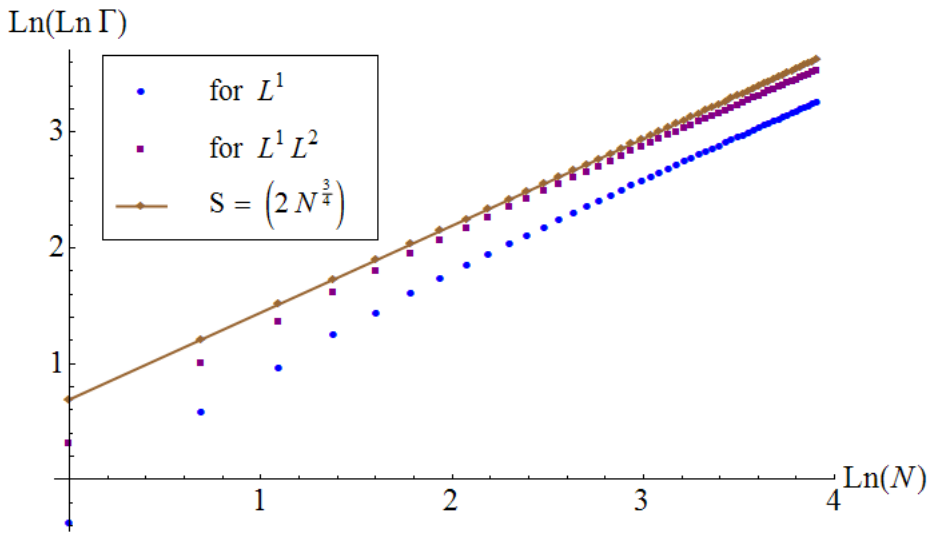

Figure 6.1: The degeneracy of the states as a function of the energy level N.

6.4.4 Eigenspace of Coupled L^1_{00}, L^2_{00} with Central Extension

Let us finally consider the full two dimensional quantum diffeomorphism algebra

$$\begin{aligned}
\left[L^1_{nm}, L^1_{kl}\right] &= (n-k)L^1_{n+k,m+l} - nk(c_1+c_2)(nS^1_{n+k,m+l} + mS^2_{n+k,m+l}), \qquad (6.42)\\
\left[L^2_{nm}, L^2_{kl}\right] &= (m-l)L^2_{n+k,m+l} - ml(c_1+c_2)(nS^1_{n+k,m+l} + mS^2_{n+k,m+l}),\\
\left[L^1_{nm}, L^2_{kl}\right] &= mL^1_{n+k,m+l} - kL^2_{n+k,m+l} - (c_1 mk + c_2 nl)(nS^1_{n+k,m+l} + mS^2_{n+k,m+l}).
\end{aligned}$$

Because of the appearance of the operators S^i_{nm}, the complete representation space of the algebra (6.42) is no longer spanned by the states (6.32).

To see this, let us first consider a state $L_{-n-m}L_{-k-l}\,|\lambda\mu\rangle$, with the ordering $n \leq k$, $m \leq l$ as explained below eq.(6.32). This is an eigenstate of the operator $A_0 = L^1_{00} + L^2_{00}$ with eigenvalue $\lambda + \mu + n + k + m + l$, and thus is one of the basis states of the highest-weight representation of (6.42).

Consider now a state

$$L_{-k-l}L_{-n-m}\,|\lambda\mu\rangle$$

for the same values of indices. The order of the indices is no longer correct, and, therefore, this is not a basis state. It is still an eigenstate of A_0 and, hence, belongs to the Verma module. This means, that it can be rewritten as a linear combination of the basis states by using the commutation relations (6.42):

$$\begin{aligned}
L_{-k-l}L_{-n-m}\,|\lambda\mu\rangle &= [L_{-k-l}, L_{-n-m}]\,|\lambda\mu\rangle + L_{-n-m}L_{-k-l}\,|\lambda\mu\rangle\\
&= (-k+n)L_{-k-n,-l-m}\,|\lambda\mu\rangle + L_{-n-m}L_{-k-l}\,|\lambda\mu\rangle\\
&\quad - kn(c_1+c_2)\left(-kS^1_{-k-n,-l-m} - lS^2_{-k-n,-l-m}\right)|\lambda\mu\rangle\,.
\end{aligned}$$

The last two terms in this expression are new and cannot be expressed in terms of states (6.32). This leads us to the conclusion that the states in (6.32) do not form a complete set of linearly independent eigenstates of the operator $A_0 = L^1_{00} + L^2_{00}$.

In order to resolve this problem, one has to check whether the states $S^i_{nm}\,|\lambda\mu\rangle$, $i = 1, 2$ are eigenstates of L^1_{00}, L^2_{00}, what are the eigenvalues, and what is the

physicality condition. Calculation of the L^1_{00} eigenvalues is straightforward

$$\begin{aligned}\left[L^1_{00}, S^1_{-n-m}\right] &= L^1_{00}S^1_{-n-m}\left|\lambda\mu\right\rangle - \lambda S^1_{-n-m}\left|\lambda\mu\right\rangle \\ &= -mS^2_{-n-m}\left|\lambda\mu\right\rangle \\ &= nS^1_{-n-m}\left|\lambda\mu\right\rangle \\ \Rightarrow L^1_{00}S^1_{-n-m}\left|\lambda\mu\right\rangle &= (\lambda+n)S^1_{-n-m}\left|\lambda\mu\right\rangle .\end{aligned}$$

Note, that explicit use of the defining relation (6.23) was made. Similar calculations allow one to write

$$\begin{aligned}L^1_{00}S^i_{-nM}\left|\lambda\mu\right\rangle &= (\lambda+n)S^i_{-nM}\left|\lambda\mu\right\rangle \\ L^1_{00}S^i_{nM}\left|\lambda\mu\right\rangle &= (\lambda-n)S^i_{nM}\left|\lambda\mu\right\rangle \\ L^2_{00}S^i_{N,-m}\left|\lambda\mu\right\rangle &= (\mu+m)S^i_{N,-m}\left|\lambda\mu\right\rangle \\ L^2_{00}S^i_{Nm}\left|\lambda\mu\right\rangle &= (\mu-m)S^i_{Nm}\left|\lambda\mu\right\rangle .\end{aligned}$$

Hence, states of the form $S^i_{nm}\left|\lambda\mu\right\rangle$ are indeed eigenstates of L^i_{00}. By anology the physicality condition reads

$$S^i_{nm}\left|\lambda\mu\right\rangle = 0, \quad n,m>0. \tag{6.43}$$

Note that we are using the superscript i instead of specifying which non-central extension is being used. This is possible since both give the same A_0 eigenvalues and commute with each other. It follows that every descendant state of the form

$$S^{i_1}_{-n_1,-m_1}\ldots S^{i_k}_{-n_k,-m_k}\left|\lambda\mu\right\rangle \quad \text{with} \quad i_r=1,2,\ n_r,m_r>0,$$

is a zero norm state. Moreover, any matrix element

$$\left\langle\lambda\mu\right|S^{i_1}_{n_1m_1}\ldots S^{i_k}_{n_km_k}\left|\lambda\mu\right\rangle = 0 \qquad \forall n_r,m_r\in\mathbb{Z}, \tag{6.44}$$

because the operators S commute and the state $\left|\lambda\mu\right\rangle$ obeys the physicality condition (6.43).

Now we would like to construct the full set of basis vectors of the representation

space of the 2D Virasoro algebra. The defining relation

$$nS^1_{nm} + mS^2_{nm} = 0 \tag{6.45}$$

suggests that S^1 and S^2 are linearly dependent. Even so, if one would like to replace all S^2_{-n-m} operators with $-\frac{n}{m}S^1_{-n-m}$ then this would correspond to negative, mostly non-integer L^1_{00} and L^2_{00} eigenvalue states. Hence, both S^i should be used independently as raising operators.

In the section 6.3 we have shown that $S^1_{n0} = \delta_{n0}S^1_{00}$ and $S^2_{0n} = \delta_{n0}S^2_{00}$. However, this is not true for the operators S^1_{0n} and S^2_{n0}. Moreover, they cannot be used to build descendant states, because this would lead to infinitely degenerate ground state. Indeed,

$$\begin{aligned}(L^1_{00} + L^2_{00})S^1_{0-n}\,|\lambda\mu\rangle &= \left([L^1_{00}, S^1_{0-n}] + [L^2_{00}, S^1_{0-n}] + (\lambda+\mu)S^1_{0-n}\right)|\lambda\mu\rangle \\ &= -nS^2_{0-n}\,|\lambda\mu\rangle + (\lambda+\mu)S^1_{0-n}\,|\lambda\mu\rangle \\ &= (\lambda+\mu)S^1_{0-n}\,|\lambda\mu\rangle\,.\end{aligned}$$

A general basis state of the Fock space is then

$$\begin{aligned}|\lambda'\mu'\rangle =& \tilde{\prod_{i=0}^{l}}(L^1_{0,-i})^{\alpha_{0i}}(L^1_{-1,-i})^{\alpha_{1i}}\ldots(L^1_{-n_i,-i})^{\alpha_{n_i i}}\tilde{\prod_{j=0}^{m}}(L^2_{0,-j})^{\beta_{0j}}(L^2_{-1,-j})^{\beta_{1j}}\ldots(L^2_{-n_j,-j})^{\beta_{n_j j}} \\ &\tilde{\prod_{p=1}^{\tilde{l}}}(S^1_{-1,-p})^{\eta_{1p}}\ldots(S^1_{-n_p,-p})^{\eta_{npp}}\tilde{\prod_{r=1}^{\tilde{m}}}(S^2_{-1,-r})^{\rho_{1rj}}\ldots(S^2_{-n_r,-r})^{\rho_{nrr}}\,|\lambda\mu\rangle\,,\end{aligned} \tag{6.46}$$

where the previous notations are used and operators are always ordered as $L^1L^2S^1S^2$. Also $\alpha_{00} = \beta_{00} = 0$, because this corresponds to the highest-weight state itself. All descendant states with the number of S operators exceeding the number of L operators are zero norm states. It follows from (6.44) and the commutation relation $[L, S] \sim S$. For further remarks let us calculate the norm of a state

$L^1_{-n-m}S^1_{-k-l}\,|\lambda\mu\rangle$:

$$\begin{aligned}
\langle\lambda\mu|S^1_{kl}L^1_{nm}L^1_{-n-m}S^1_{-k-l}\,|\lambda\mu\rangle &= \\
&= \langle\lambda\mu|\, S^1_{kl}\left([L^1_{nm},L^1_{-n-m}]+L^1_{-n-m}L^1_{nm}\right)S^1_{-k-l}\,|\lambda\mu\rangle \\
&= \langle\lambda\mu|\, S^1_{kl}\left(2nL^1_{00}+n^2(c_1+c_2)(nS^1_{00}+mS^2_{00})\right)S^1_{-k-l}\,|\lambda\mu\rangle + \langle\lambda\mu|\, S^1_{kl}L^1_{-n-m}[L^1_{nm},S^1_{-k-l}]\,|\lambda\mu\rangle \\
&= 2n\,\langle\lambda\mu|\, S^1_{kl}\left([L^1_{00},S^1_{-k-l}]+S^1_{-k-l}L^1_{00}\right)|\lambda\mu\rangle - l\,\langle\lambda\mu|\, S^1_{kl}L^1_{-n-m}S^2_{n-k,m-l}\,|\lambda\mu\rangle \\
&= -l\,\langle\lambda\mu|\,[S^1_{kl},L^1_{-n-m}]S^2_{n-k,m-l}\,|\lambda\mu\rangle \\
&= l^2\,\langle\lambda\mu|\, S^2_{k-n,l-m}S^2_{n-k,m-l}\,|\lambda\mu\rangle\,.
\end{aligned}$$

This is only non-zero if $n=k$ and $m=l$. Therefore we conclude that only the states with the total sum of the first (second) indices of L operators being larger [1] than the sum of the corresponding indices of S operators can have non-zero norm. The L^1_{00} eigenvalue N_1 of the state (6.46) is the sum of the first indices of all the raising operators. To see this let us calculate the L^1_{00} eigenvalue of the state $|\phi\rangle \equiv L^1_{-n-m}L^2_{-k-l}S^1_{-g-f}\,|\lambda\mu\rangle$ as an example

$$\begin{aligned}
L^1_{00}\,|\phi\rangle &= \left[L^1_{00},L^1_{-n-m}L^2_{-k-l}S^1_{-f-g}\right]|\lambda\mu\rangle + \lambda\,|\phi\rangle \\
&= L^1_{-n-m}L^2_{-k-l}\left[L^1_{00},S^1_{-f-g}\right]|\lambda\mu\rangle + \left[L^1_{00},L^1_{-n-m}L^2_{-k-l}\right]S^1_{-f-g}\,|\lambda\mu\rangle + \lambda\,|\phi\rangle \\
&= L^1_{-n-m}L^2_{-k-l}(-g)S^2_{-f-g}\,|\lambda\mu\rangle + L^1_{-n-m}\left[L^1_{00},L^2_{-k-l}\right]S^1_{-f-g}\,|\lambda\mu\rangle \\
&\quad + \left[L^1_{00},L^1_{-n-m}\right]L^2_{-k-l}S^1_{-f-g}\,|\lambda\mu\rangle + \lambda\,|\phi\rangle \\
&= (\lambda+n+k+f)\,|\phi\rangle\,.
\end{aligned}$$

The degeneracy of the $L^1_{00}+L^2_{00}$ eigenvalue thus increases. However, by analogy with the previous chapter, we conclude that the asymptotics still remain as in Figure 6.1.

6.5 Speculations

In this section we would like to point out some analogies between the Fock space spanned by the states (6.46) and the Fock space of the string theory spanned by (3.67). We will then reveal the difficulties arising in the case of two dimensional diffeomorphisms.

[1] Indices n, m are positive integers

Let us start with rewriting the state (6.46) in the form:

$$|\{\alpha,\beta\},\lambda\mu\rangle = \prod_{n,m=0}^{\infty} (L^1_{-n-m})^{\alpha^1_{nm}} (L^2_{-n-m})^{\alpha^2_{nm}} (S^1_{-n-m})^{\beta^1_{nm}} (S^2_{-n-m})^{\beta^2_{nm}} |\lambda\mu\rangle \,. \quad (6.47)$$

Here $\{\alpha\} = \{\alpha^1, \alpha^2\}$ and $\{\beta\} = \{\beta^1, \beta^2\}$ are strings of non-negative integers and $\alpha^1_{00} = \alpha^2_{00} = \beta^1_{00} = \beta^2_{00} = 0$. Comparison with equations (3.67) and (3.72) together with commutation relations (3.65) and (6.22) leads to several observations. First, the operators S and the operators K of string theory obey very similar commutation relations with diffeomorphism generators and they both appear in the basis states of the corresponding Fock spaces. We know that the operators K were crucial for proving that the states (3.67) are indeed linearly independent. With appropriate ordering we were able to bring the inner product matrix $\mathcal{M}^P$ in the form (3.70), which ensured that $\det(\mathcal{M}^P) \neq 0$. Therefore one could ask whether the same procedure can be performed here, but with operators S "playing the role" of the operators K? Without going into details, we just mention that it is possible to define an ordering of states (6.47) similar to the ordering, which was used in string theory [17]. However, several elements on the minor diagonal are zero as a consequence of the fact that $S^1_{n0} = S^2_{0n} = 0$ if $n \neq 0$. This property of S operators followed from the defining relation $nS^1_{nm} + mS^2_{nm} = 0$, which is not present in the 1D case. Hence, the determinant of the two dimensional analogue of the ordered matrix $\mathcal{M}^P$ is zero, and we conclude that the operators S cannot be associated to the operators K.

Second, we note that in the description of closed two dimensional spacelike surfaces the physical DDF states of string theory are substituted with the highest-weight states $|\lambda\mu\rangle$. While the construction of DDF states is clear and was presented in section 3.6.3, the physical origin of the states $|\lambda\mu\rangle$ is unknown. In string theory it was shown that the physical states are built with the negative frequency modes of the transverse oscillators α^I_{-n}. However, there is no such analogue in general relativity.

One might exploit the fact that the transverse Virasoro operators are expressed

via oscillators as

$$L_n^{\perp} = \frac{1}{2}\sum_m \alpha^I_{n-m}\alpha^I_m, \qquad \text{with}$$
$$[\alpha^I_n, \alpha^I_m] = n\delta_{n+m,0},$$

and try to find an analogous expansion of operators L^1_{nm}, L^2_{nm}. Indeed, if one defines

$$L^1_{nm} = \frac{1}{2}\sum_{k,l} J^1_{n-k,m-l}J^1_{kl}, \tag{6.48}$$
$$L^2_{nm} = \frac{1}{2}\sum_{k,l} J^2_{kl}J^2_{n-k,m-l}$$

with

$$[J^1_{nm}, J^1_{kl}] = n\delta_{n+k,0}\delta_{m+l,0},$$
$$[J^2_{nm}, J^2_{kl}] = m\delta_{n+k,0}\delta_{m+l,0},$$

then it satisfies the commutation relations $[L^1_{nm}, L^1_{kl}]$, $[L^2_{nm}, L^2_{kl}]$ in (6.29). However, no expansion which would satisfy the mutual commutator $[L^1_{nm}, L^2_{kl}]$ was found. The difficulties arise when we are defining the commutation relation $[J^1_{nm}, J^2_{kl}]$. If it is zero, then the operators L^1 and L^2 commute. If it is a c-number, then the commutator becomes

$$[L^1, L^2] \sim J^1 J^2.$$

Such a mixed term is absent in the definitions (6.48). Thus, either the commutator $[J^1_{nm}, J^2_{kl}]$ gives some operator, or the expansions (6.48) have to be modified. However, none of the possibilities is trivial, and therefore there is no obvious expansion of the operators L^i_{nm}. If there would exist a construction of the generators of the 2d diffeomorphisms in terms of operators J^i, which form an affine Lie algebra, then the operators could, by analogy with string theory, be used to generate physical states.

In such a case there would still remain one unanswered question, namely, whether the states like

$$|\phi\rangle = J^i_{-k-l}\,|0\rangle$$

would obey the gauge invariance condition

$$L^i_{nm} |\phi\rangle = 0, \quad \forall n, m > 0?$$

Recall that in string theory $[L^\perp_n, \alpha^I_{-k}] = k\alpha^I_{n-k}$ and thus

$$L^\perp_n \left(\alpha^I_{-k} |0\rangle\right) = k\alpha^I_{n-k} |0\rangle = 0, \qquad \text{if} \quad n \geq k.$$

Hence, the operator $L^\perp_n$ does not "kill" the state $\alpha^I_{-k} |0\rangle$ in the general case. The reason why a state built from the transverse oscillators is physical, is that in the light-cone gauge quantization the constraints of conformal invariance $L = \bar{L} = 0$ are solved explicitly, and thus have no longer to be imposed on the quantum states built with the raising operators α^I_{-n}. In general relativity, however, it is not possible to resolve the constraint equations explicitly, because these are very non-linear. On the other hand, the issue with non-physical states in string theory arose only due to the fact that the C_0 constraint was used in the definitions of the operators L and $\bar{L}$. This, combined with the commutation relation $[\alpha^0_n, \alpha^0_m] = -n\delta_{n+m,0}$ lead to negative-norm states. The Virasoro constraints were then used to single out these negative-norm states. As in the description of 2d spacelike surfaces only the constraints $\mathcal{H}_1$ and $\mathcal{H}_2$ were used, we believe that there should be no ghost states among the states, generated by the "transverse" raising operators J^1 and J^2.

6.6 Possible Relation with the Quantization of Black Holes

In sections 5.4 and 5.5 an algebraic description of black holes was presented as it was proposed by Bekenstein [5]. It follows that a nonrotating black hole can be described by a closed set of operators $\{A, Q, R_{nqs}\}$, where A is the area operator with eigenvalues a_n, Q is the generator of gauge transformations with eigenvalues labeled by q, and R_{nqs} is the creation operator of a black hole state $R_{nqs} |0\rangle$ with eigenvalues a_n, q, and s, where s is some internal quantum number accounting for the degeneracy of a given area eigenvalue. These operators satisfy the following

properties:

$$[A, R_{nqs}] = a_n R_{nqs}, \tag{6.49}$$

$$[A, R^{\dagger}_{nqs}] = -a_n R_{nqs}, \tag{6.50}$$

$$AR_{nqs}R_{m\tilde{q}t}\left|0\right\rangle = (a_n + a_m)R_{nqs}R_{m\tilde{q}t}\left|0\right\rangle, \tag{6.51}$$

$$[A, Q] = 0, \tag{6.52}$$

$$[Q, R_{nqs}] = qR_{nqs}. \tag{6.53}$$

Among the operators in the two dimensional diffeomorphism algebra, there are two kinds of operators which could be of relevance for the quantum algebra of a black hole. First, consider the operators L^i_{nm} with n and m not equal to zero simultaneously. From the commutation relations (6.31) it follows that these can be interpreted as the raising and lowering operators of the eigenvalues of the operator $A_0 = L^1_{00} + L^2_{00}$. Hence, the properties of the operators A_0 and L^i_{nm} are very similar to (6.49), (6.50), and (6.51). This suggests that we identify the operator A_0 as the area operator and the operators L^i_{nm} as the creation-annihilation operators of the black hole states.

On the other hand, we know that the operators L^i_{nm} are the generators of the 2d diffeomorphism transformations, and thus should be instead associated to the operators Q, which account for the gauge transformations. Hence, it seems that the operators L^i_{nm} should be used as constraints for identifying physical states which differ from each other only by a diffeomorphism transformation, instead of creating and annihilating states that correspond to different values of the black hole horizon area. However, in analogy with the light-cone gauge quantization in string theory we could say that the $\mathcal{H}_0$ and $\mathcal{H}_3$ constraints impose the physicality conditions, while the $\mathcal{H}_1$ and $\mathcal{H}_2$ constraints generate the transformations between different *physical* states. This is, however, not a solid statement. Nevertheless, we suggest the identification of the operatos A_0 with the area operator and the operators L^i_{nm} with the "transverse" creation and annihilation operators.

7

Conclusions

In this work we have explored the role of diffeomorphism invariance in general relativity. The possible relation between distinct physical black hole microstates corresponding to a given area eigenvalue and the quantum states, built by successive application of the generators of diffeomorphism transformations on the vacuum state, was investigated. We have, first, reviewed the use of the one dimensional diffeomorphism algebra (Virasoro algebra) in string theory and compared it to that of conformal field theory. The emphasis was laid on the construction of physical state space in string theory and the highest-weight representation space in conformal field theory.

We have, further, been looking for a physical justification of the operators which generate the quantum states of a black hole. The possible relevance of quantum generators of spatial diffeomorphism transformations to the area quantization of spacelike surfaces was considered. Two of the spatial constraints of general relativity were expanded in a Fourier series on a surface with the topology of a torus. The properties of the resulting two dimensional diffeomorphism algebra were explored. It was shown that the 2d diffeomorphism algebra does not have a central extension. Instead, the algebra has a non-central extension, which is given by a set of operators that were previously absent.

We have found that this algebra can be characterized by highest-weight representations, similarily as it was done for the 1d Virasoro algebra in conformal field theory. The highest-weight representations of the 2d diffeomorphism algebra, consisting of the operators L^1_{nm} and L^2_{nm}, were considered, first, without including the

non-central extensions. It was then shown that the degeneracy of a given eigenvalue of the diagonal operator $A_0 = L^1_{00} + L^2_{00}$ is increased in comparison with the degeneracy of $L_0 + \bar{L}_0$ eigenvalues in string and conformal field theories.

Further, the properties of the operators which generate the non-central extension of the 2d diffeomorphism algebra were explored. It was shown that these contribute as raising and lowering operators of the highest-weight representation. Therefore, the Hilbert space which corresponds to the 2d diffeomorphism algebra with non-central extensions is larger than in the case without extensions. Hence, the degeneracy of an eigenvalue of the operator A_0 also increases.

The operator A_0 can be associated with the area operator, as it seems that it obeys its properties, as listed by Bekenstein [5]. However, not all of the descendant states, built by repeated action with the raising operators on the highest weight state, are physical states. Some of them do not satisfy the physicality condition $L^i_{nm} |\phi\rangle = 0, \forall n, m > 0$. Instead, some of the descendants of a physical state $|\phi\rangle$ are the states which differ from it by a 2d diffeomorphism transformation. Further investigation to understand how the physical states of a closed surface could be produced is needed. The properly modified $\mathcal{H}_0$ constraint might happen to be necessary, as it is responsible for generating dynamics in general relativity [32], and further work in this direction should be done.

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Erklärung

Hiermit versicher ich an Eides statt, diese Arbeit selbständig angefertigt zu haben und keine anderen als die angegebenen Hilfsmittel verwendet zu haben.

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