Elektrischer Quadrupol

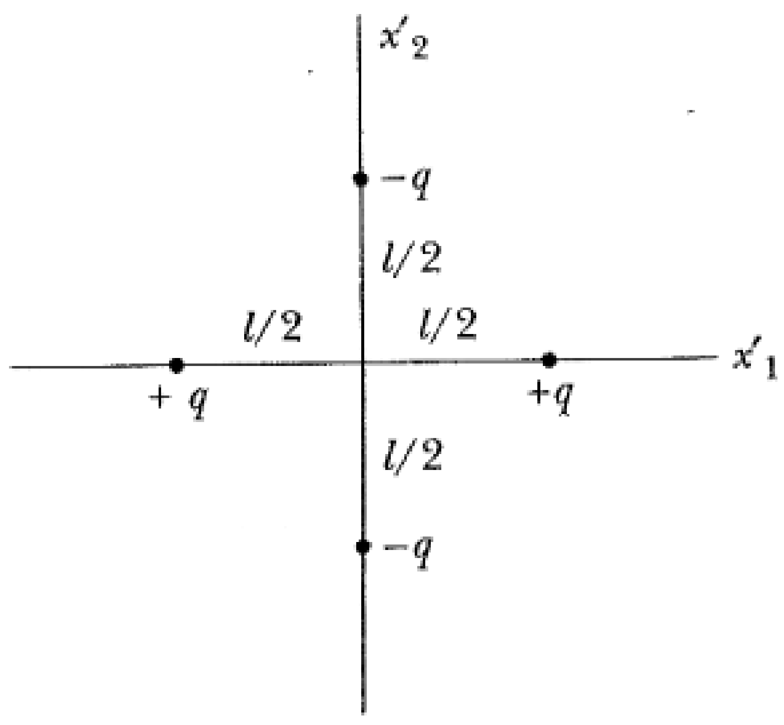


FIGURE 2-6. A square quadrupole.

where $\varphi = 0$ along the positive x_1 axis. The potential in the x_1-x_2 plane ($\theta = \pi/2$) is shown in Fig. 2-7; again there are both positive and negative portions of the potential.

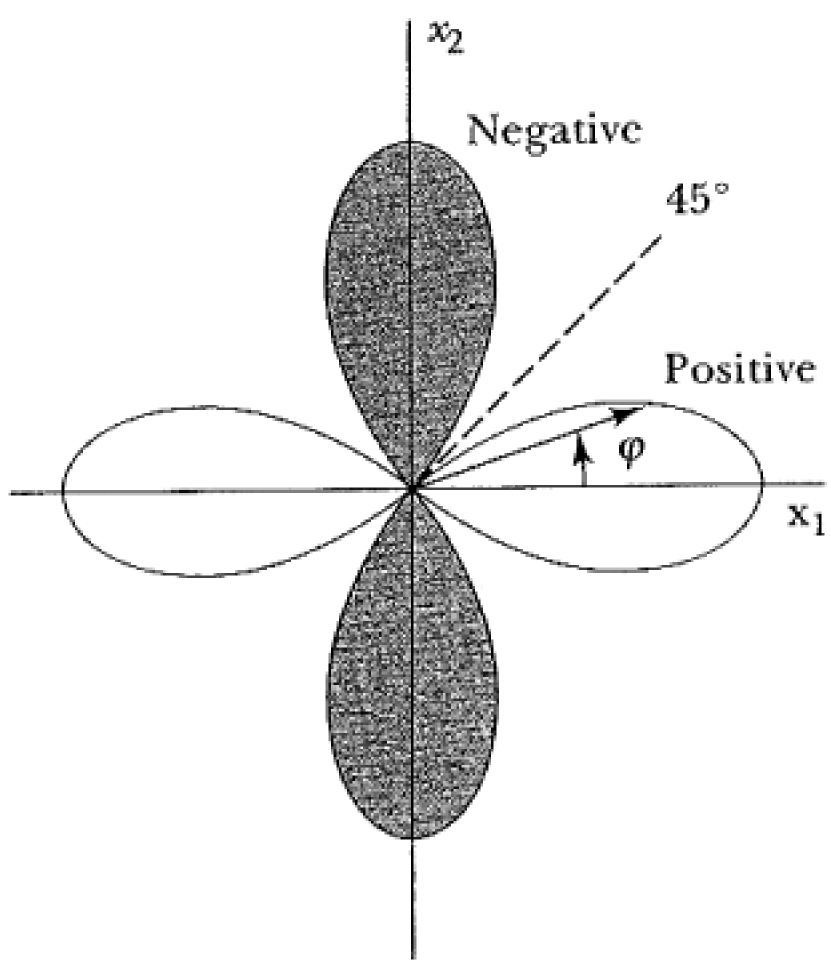


FIGURE 2-7. Polar plot of potential.

$$\phi = \frac{3}{4} q l^2 \frac{1}{r^3} \sin^2\theta \cos 2\varphi$$

Elektr. Quadrupol

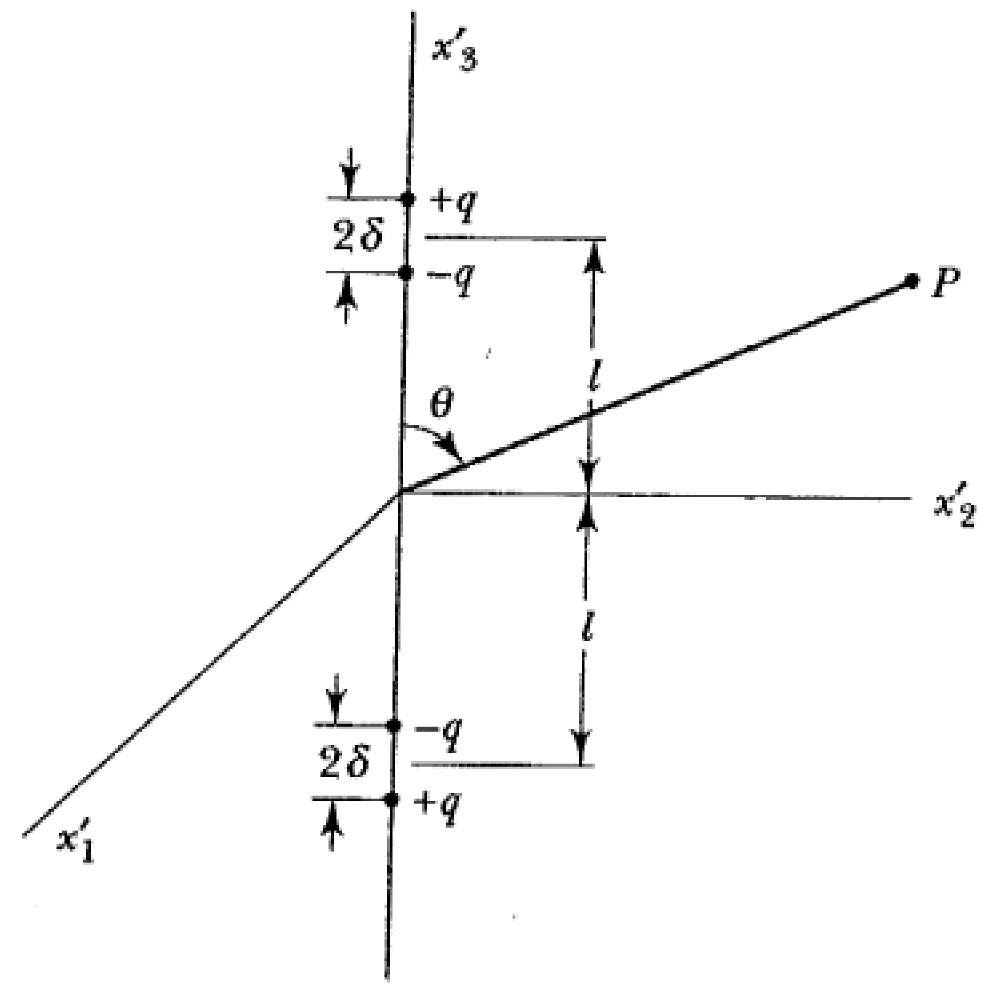


FIGURE 2-4. An axial quadrupole.

$$Q_{11} = Q_{22} = -\frac{1}{2} Q_{33} = -4pl$$

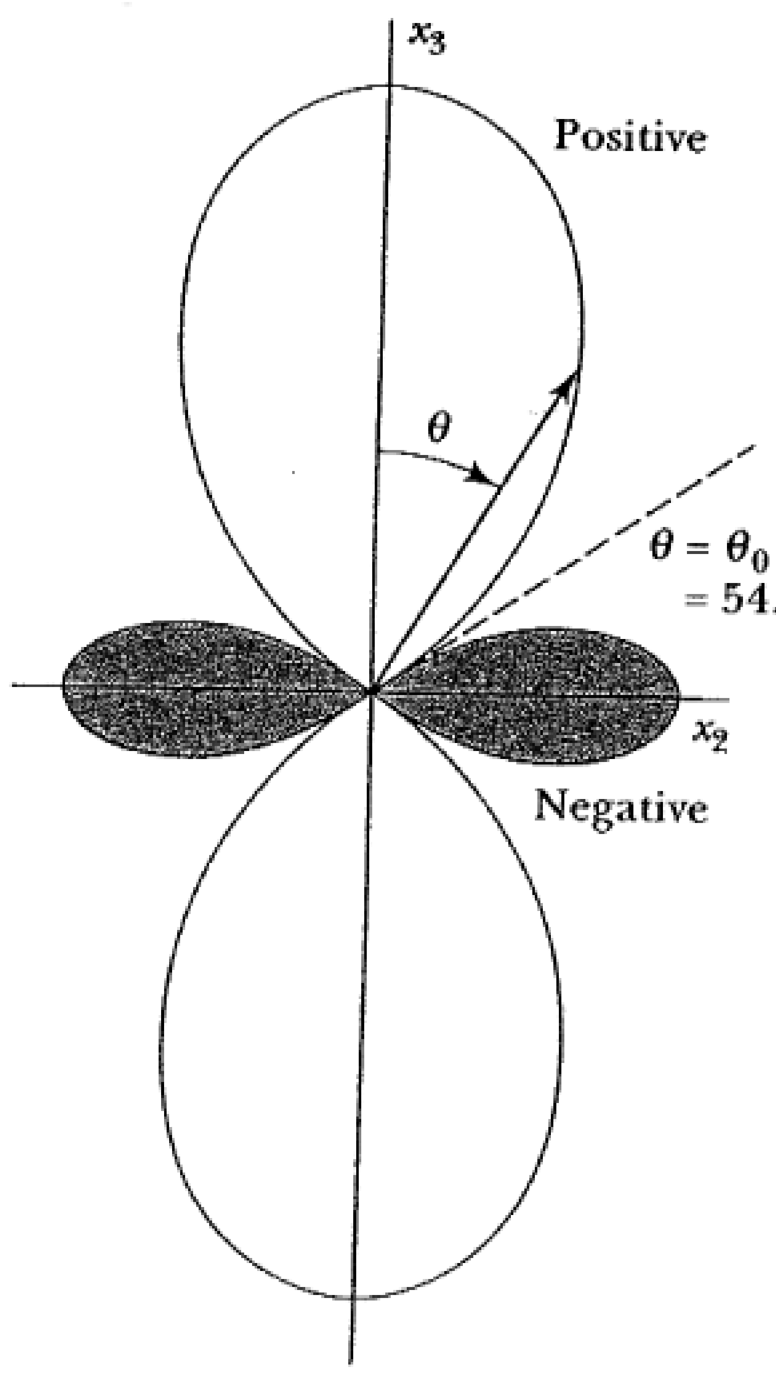


FIGURE 2-5. Polar plot of quadrupole potential.

$$\phi = 2pl \, \frac{3\cos^2\theta - 1}{r^3}$$

Eine formale Entwicklung nach Multipolen findet man über Kugelflächenfunktionen

$$\varphi(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} Y_{lm}(\theta, \phi)\, q_{lm}$$

wobei $q_{lm} = \int d^3\vec{x}'\, |\vec{x}'|^l\, Y^*_{lm}(\theta', \phi')\, \rho(\vec{x}')$

die Multipolmomente der Ladungsverteilung sind.

Es gilt $q_{l,-m} = (-1)^m q^*_{l,m}$

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(\vec{y})\, d^3y = \frac{1}{\sqrt{4\pi}} Q$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int d^3y\, \rho(\vec{y}) |\vec{y}| \sin\theta (\cos\phi - i \sin\phi)$$

$$= -\sqrt{\frac{3}{8\pi}} (p_x - i p_y)$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int d^3y\, \rho(\vec{y})\, |\vec{y}| \cos\theta = \sqrt{\frac{3}{4\pi}}\, p_z$$

$$q_{1,-1} = -q^*_{11} = \sqrt{\frac{3}{8\pi}} (p_x + i p_y)$$

$l = 1 \quad \hat{=} \quad$ Dipol

Entsprechend erhält man aus $l = 2$ die Quadrupolmomente.

(b) Magnetische Multipole

Wir gehen von der Poissonformel für das Vektorpotential aus

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3\vec{x}' \; \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

und führen wieder eine Taylorentwicklung von $|\vec{x}-\vec{x}'|^{-1}$ durch

$$\vec{A}(\vec{x}) = \frac{1}{cr}\int d^3\vec{x}' \; \vec{j}(\vec{x}')$$

$$+ \frac{1}{cr^3}\int d^3\vec{x}' \; (\vec{x}\cdot\vec{x}') \; \vec{j}(\vec{x}') + \ldots$$

1. Term:

$$\int d^3\vec{x}' \, j_k(\vec{x}') = \int d^3\vec{x}' \; \overbrace{\frac{\partial x_k}{\partial x_l'}}^{=\delta_{kl} \text{ (Summen-konvention)}} j_l(\vec{x}')$$

$$\underset{\text{p.I.}}{=} -\int d^3\vec{x}' \; x_k' \underbrace{\frac{\partial j_l(\vec{x}')}{\partial x_l'}}_{\substack{= \mathrm{div}'\vec{j}(\vec{x}') \\ \underset{\text{Kontinuitätsgl.}}{=} -\partial_t \rho \underset{\text{stat. Fall}}{=} 0}}$$

$$= 0.$$

Beachte, falls $\partial_t \rho \neq 0$ aber monochrom, d.h. $-\partial_t \rho = i\omega\rho$, dann

$$\int d^3\vec{x}' \; \vec{j}(\vec{x}') = -i\omega \int d^3\vec{x}' \; \vec{x}' \rho(\vec{x}') = -i\omega\vec{p}.$$

Das brauchen wir später noch!

2. Term

$$\frac{1}{r^3 c}\, x_l \underbrace{\int d^3x'\, x'_l\, j_k(\vec{x}')}_{\frac{1}{2}\int\left(\left[x'_l j_k + x'_k j_l\right] + \left[x'_l j_k - x'_k j_l\right]\right) d^3x'}$$

Symmetrischer Anteil

$$\int (x'_l j_k + x'_k j_l)\, d^3x' =$$

$$= \int\left((\partial'_m x'_k)\, x'_l\, j_m + (\partial'_m x'_l)\, x'_k\, j_m\right) d^3x' =$$

$$\underset{p.I.}{=} -\int\left[x'_k\left(\delta_{lm} j_m + x'_l\, \vec{\nabla}'\cdot\vec{j}\right) + x'_l\left(\delta_{mk} j_m + x'_k(\vec{\nabla}'\cdot\vec{j})\right)\right] d^3x'$$

$$= -\int (x'_k j_l + x'_l j_k)\, d^3x' - 2\int x'_l x'_k (\vec{\nabla}'\cdot\vec{j})\, d^3x'$$

$$\Longrightarrow \int (x'_k j_l + x'_l j_k)\, d^3x' = -\int x'_l x'_k (\vec{\nabla}'\vec{j})\, d^3x'$$

$$= -i\omega \underbrace{\int x'_l x'_k\, \rho(\vec{x}')\, d^3x'}_{Q'_{kl}}$$

Anti-symm. Anteil

$$\frac{1}{2c}\int d^3x' \Big[\underbrace{(\vec{x}\vec{x}')\,\vec{j}(\vec{x}') - (\vec{x}\cdot\vec{j}(\vec{x}'))\,\vec{x}'}_{} \Big]$$

$$= \left[\vec{x}' \times \vec{j}(\vec{x}')\right] \times \vec{x}$$

$$= \vec{m} \times \vec{x}$$

wobei das magnetische Moment definiert ist als

$$\vec{m} = \frac{1}{2c} \int d^3\vec{x}' \left(\vec{x}' \times \vec{j}(\vec{x}') \right)$$

Damit ergibt sich insgesamt

$$\vec{A}(\vec{x}) = -\frac{i\omega}{c} \frac{\vec{p}}{r} - \frac{i\omega}{r^3 c} \frac{1}{2} Q'_{kl} x_l + \frac{\vec{m} \times \vec{x}}{r^3}$$

wobei im statischen Fall nur der letzte Term vorhanden ist.

Daraus läßt sich nun leicht das magn. Feld eines Dipols berechnen

$$\vec{A} = \frac{\vec{m} \times \vec{x}}{r^3} = -\vec{m} \times \vec{\nabla} \frac{1}{r}$$

$$= \vec{\nabla} \frac{1}{r} \times \vec{m} = \vec{\nabla} \times \frac{\vec{m}}{r}$$

$$\vec{B} = \mathrm{rot}\, \vec{A} = \mathrm{rot}\, \mathrm{rot} \frac{\vec{m}}{r}$$

$$= \mathrm{grad}\, \mathrm{div} \frac{\vec{m}}{r} - \Delta \frac{\vec{m}}{r}$$

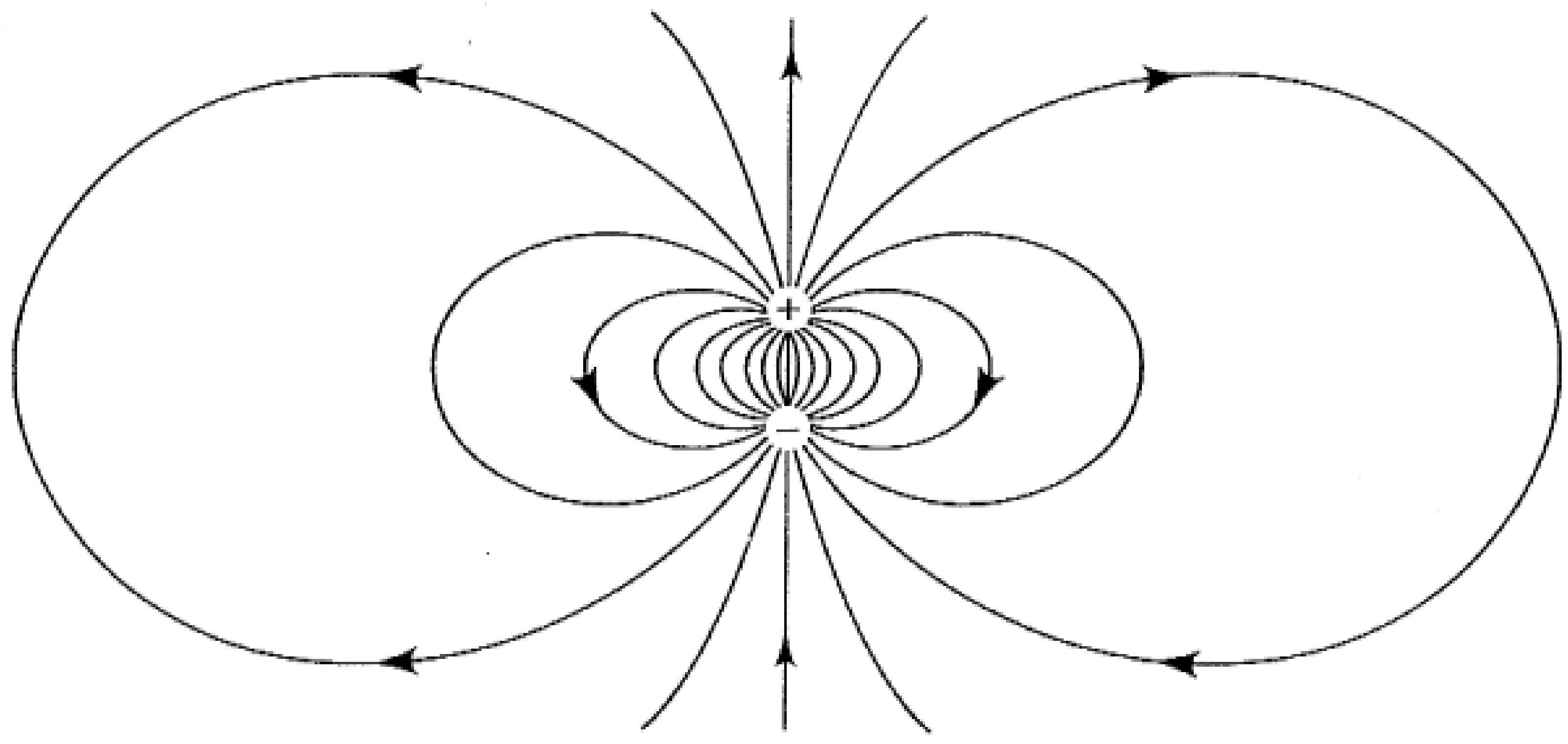
$$= -\mathrm{grad} \frac{\vec{m} \cdot \vec{x}}{r^3} - \vec{m} \Delta \frac{1}{r}$$

$$= \frac{1}{r^5} \left[3(\vec{m} \cdot \vec{x}) \vec{x} - \vec{m} r^2 \right]$$

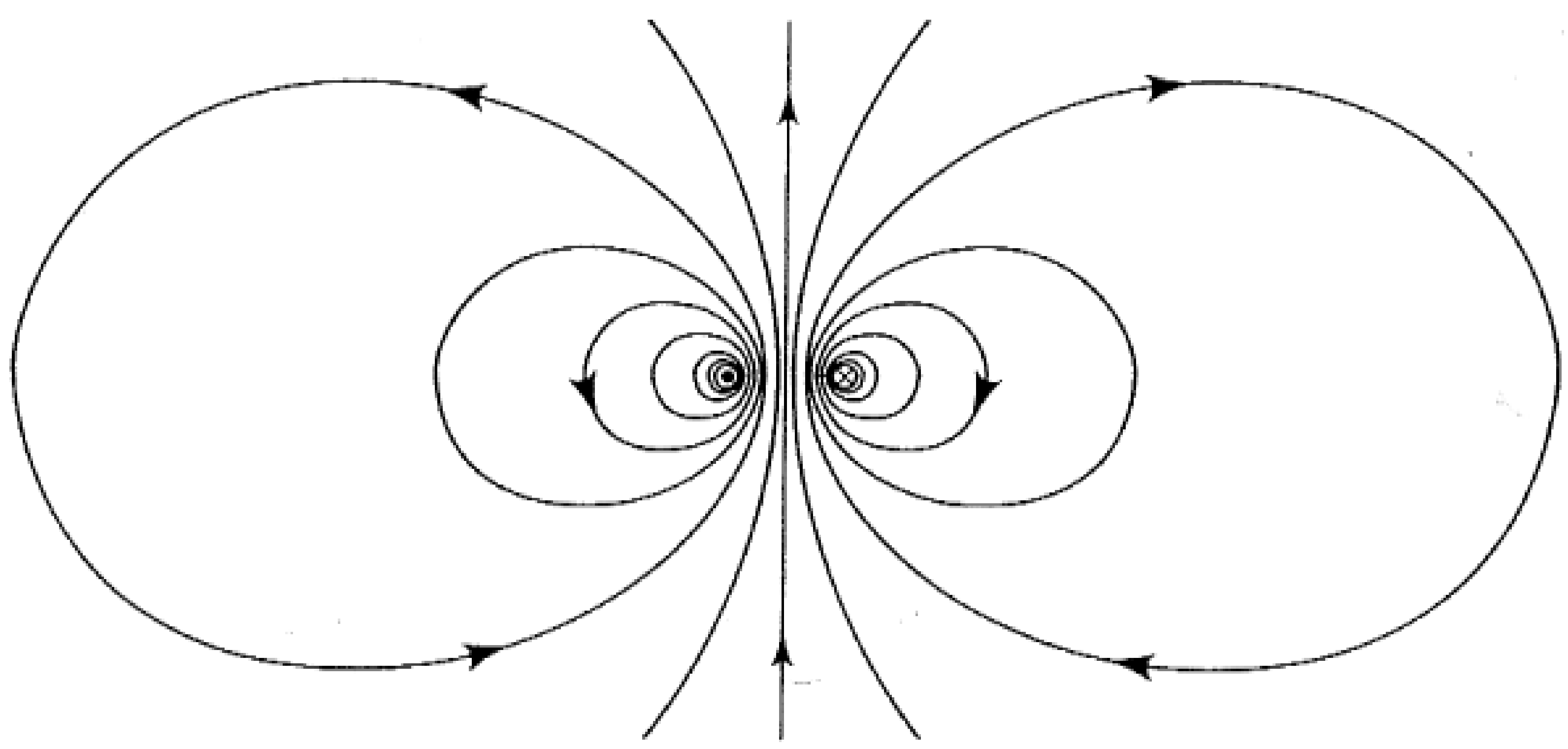
hat genau die gleiche Form wie ein elektrischer Dipol

$$\vec{B} = \frac{m}{r^3}\left(2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta\right)$$

In both the electric and magnetic cases, Eqs. (2.28) and (2.63), we treated the limit where the structure size of the dipole is negligibly small—that is, it is their *external* fields that turn out to be identical. If we look *inside* the dipole structure, however, the fields are vastly different.* Typical examples are shown in Fig. 2-11. In both cases the internal fields are very strong—but the internal field of the electric (charge-pair) case is *opposite* to the dipole-moment vector, while that of the magnetic (current-loop) case is in the *same sense* as the moment.



(a) Electric (charge-pair) dipole



(b) Magnetic (current-loop) dipole

FIGURE 2-11. Field-lines of dipole models of finite size, showing identical external fields but oppositely directed internal fields. (Both diagrams are figures-of-revolution about the vertical axis.)

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*There is a subtlety in properly representing the field *within* a "point" dipole. For some purposes it is necessary to add a delta-function to Eqs. (2.28) and (2.63), and the coefficient and sign of this term are *different* in the electric and magnetic cases. See Problem 2-19, and Jackson (Ja75, Eqs. 4.20 and 5.64). *Outside* the dipole, of course, this term is irrelevant. The similarities and differences between electric and magnetic dipoles are considered further in the following section. For further mathematical subtleties of point-source fields, see Bowen, *Am. J. Phys.* **62,** 511 (1994).

Aus Heald and Marion,
Classical Electromagnetic Radiation