VECTOR RELATIONS IN CURVILINEAR COORDINATES

A list of vector operators expressed in the common orthogonal curvilinear coordinates is often useful in the solution of physical problems. For the derivation of these relations, it is possible to procede quite formally from the definition of the operator ∇ in Cartesian coordinates and the transformation equations to other coordinate systems, but for physical applications it is advantageous to work from the geometrical definitions of gradient, divergence, and curl. One may first specify the coordinate system and derive the required expressions, or make a general derivation valid for any curvilinear coordinates and only then specify the coordinates. We shall follow the latter plan, first outlining a derivation valid for any right-handed system of orthogonal coordinates for which the line element is known, and then writing the particular forms for Cartesian, cylindrical, and spherical polar coordinates.

A line element in three dimensions is an infinitesimal displacement in space. If only one of three orthogonal coordinates q_1, q_2, q_3 is varied, the corresponding line element may be written

$$ds_1 = h_1\, dq_1,$$

(1)

together with similar expressions in q_2 and q_3. For any infinitesimal displacement,

$$ds^2 = h_1^2\, dq_1^2 + h_2^2\, dq_2^2 + h_3^2\, dq_3^2.$$

(2)

Now the gradient of a scalar function ψ is defined by the requirement that

$$\nabla\psi \cdot d\mathbf{s} = d\psi,$$

(3)

giving the change in ψ corresponding to the space displacement $d\mathbf{s}$. Then

$$(\nabla\psi)_1 = \lim_{ds_1 \to 0} \frac{\psi(q_1 + dq_1) - \psi(q_1)}{ds_1} = \frac{1}{h_1} \frac{\partial\psi}{\partial q_1}$$

(4)

is the general form of a gradient component.

To find the divergence, we shall consider an infinitesimal volume $dv = ds_1\, ds_2\, ds_3$ bounded by the surfaces $q_1 = $ constant, $q_1 + dq_1 = $ constant, etc., as indicated in Fig. III–1. Let us apply Gauss's divergence theorem, $\int \nabla \cdot \mathbf{A}\, dv = \int \mathbf{A} \cdot d\mathbf{S}$, to a vector $\mathbf{A}(q_1, q_2, q_3)$ with components A_1, A_2, A_3, integrating over this infinitesimal volume. The integral of the outward normal component of $\mathbf{A}$ over the two surfaces
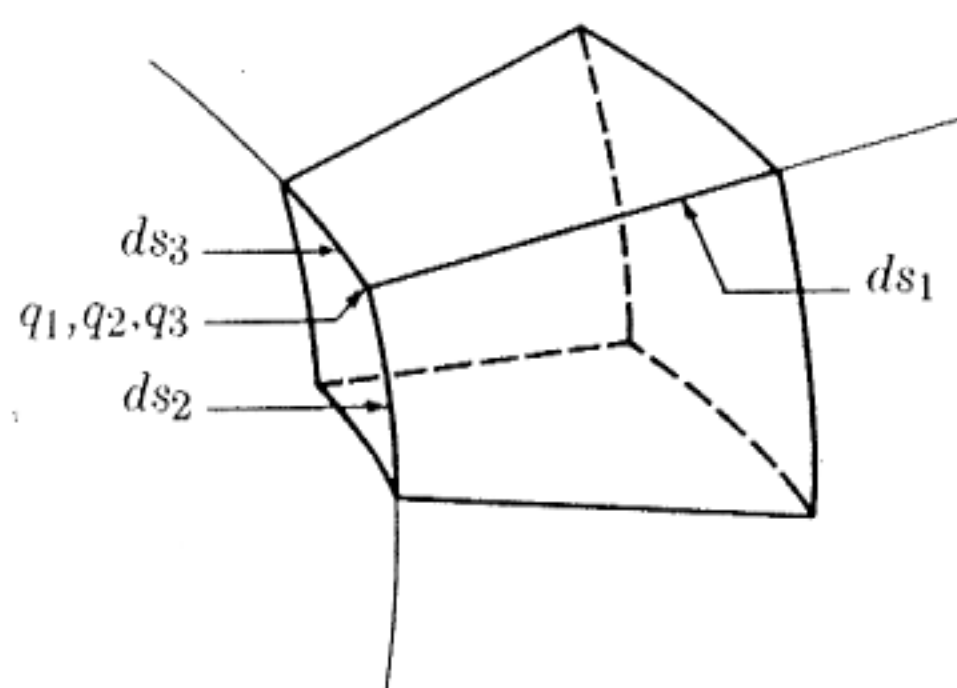
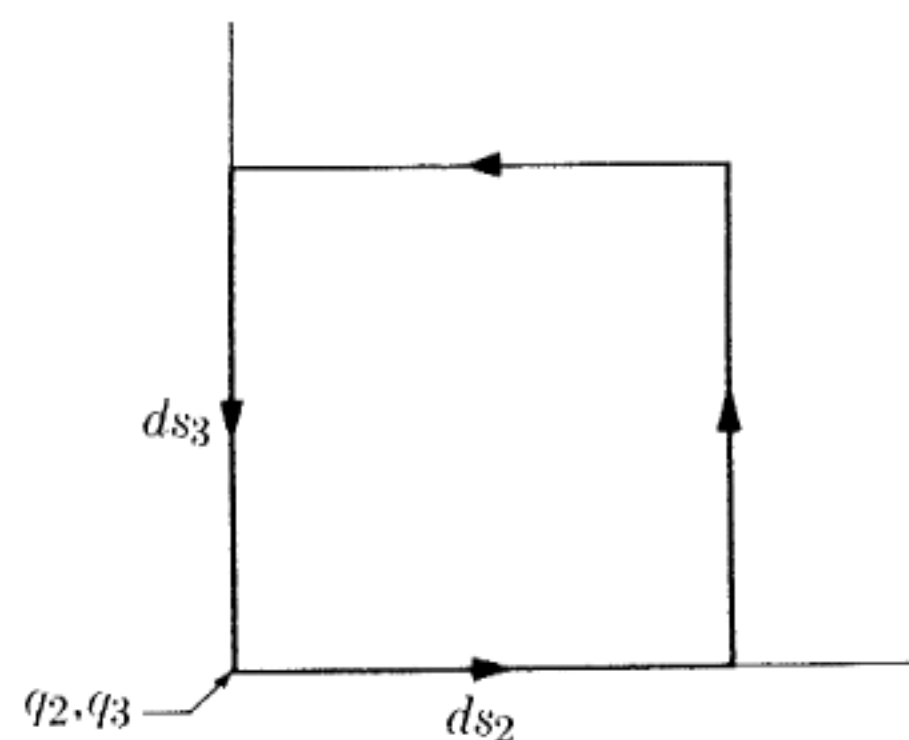FIG. III–1. Element of volume for computing the divergence.

FIG. III–2. Element of area in the q_2, q_3 plane for finding the q_1 component of the curl. Arrows show the direction of the path of integration.

perpendicular to the direction of increasing q_1 is

$$(A_1 \, ds_2 \, ds_3)_{q_1+dq_1} - (A_1 \, ds_2 \, ds_3)_{q_1} = \frac{\partial}{\partial q_1} (A_1 \, ds_2 \, ds_3) \, dq_1$$

$$= \frac{\partial(h_2 h_3 A_1)}{\partial q_1} \, dq_1 \, dq_2 \, dq_3$$

$$= \frac{1}{h_1 h_2 h_3} \frac{\partial(h_2 h_3 A_1)}{\partial q_1} \, dv,$$

and analogous expressions hold for the other two sets of surfaces. Since the sum of these three terms is, by Gauss's theorem, equal to $\nabla \cdot \mathbf{A} \, dv$, the divergence is given explicitly by

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial(h_2 h_3 A_1)}{\partial q_1} + \frac{\partial(h_3 h_1 A_2)}{\partial q_2} + \frac{\partial(h_1 h_2 A_3)}{\partial q_3} \right). \tag{5}$$

The Laplacian of a scalar function can be written down immediately, since it is just the divergence of the gradient:

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right]. \tag{6}$$

To obtain a particular component of the curl of a vector, we may apply Stokes' theorem to an infinitesimal area at right angles to the direction of the desired component. Consider the area defined by ds_2 and ds_3, as in Fig. III–2. By Stokes' theorem, $\oint \mathbf{A} \cdot d\mathbf{s} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$, which, in this application, becomes

$$(A_2 \, ds_2)_{q_3} + (A_3 \, ds_3)_{q_2+dq_2} - (A_2 \, ds_2)_{q_3+dq_3} - (A_3 \, ds_3)_{q_2}$$
$$= (\nabla \times \mathbf{A})_1 \, ds_2 \, ds_3.$$

TABLE III-1 COORDINATE SYSTEMS

Cartesian coordinates	Cylindrical coordinates	Spherical polar coordinates
	Orthogonal line elements	
dx, dy, dz	$dr, r\, d\varphi, dz$	$dr, r\, d\theta, r \sin\theta\, d\varphi$
	Components of gradient	
$(\nabla\psi)_x = \dfrac{\partial\psi}{\partial x}$	$(\nabla\psi)_r = \dfrac{\partial\psi}{\partial r}$	$(\nabla\psi)_r = \dfrac{\partial\psi}{\partial r}$
$(\nabla\psi)_y = \dfrac{\partial\psi}{\partial y}$	$(\nabla\psi)_\varphi = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\varphi}$	$(\nabla\psi)_\theta = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$
$(\nabla\psi)_z = \dfrac{\partial\psi}{\partial z}$	$(\nabla\psi)_z = \dfrac{\partial\psi}{\partial z}$	$(\nabla\psi)_\varphi = \dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial\varphi}$
	The divergence of $\mathbf{A} = \nabla\cdot\mathbf{A}$	
$\dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{r}\dfrac{\partial(r A_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial A_\varphi}{\partial\varphi} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{r^2}\dfrac{\partial(r^2 A_r)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial(\sin\theta\, A_\theta)}{\partial\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial A_\varphi}{\partial\varphi}$
	Components of curl $\mathbf{A}$	
$(\nabla\times\mathbf{A})_x = \left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right)$	$(\nabla\times\mathbf{A})_r = \left(\dfrac{1}{r}\dfrac{\partial A_z}{\partial\varphi} - \dfrac{\partial A_\varphi}{\partial z}\right)$	$(\nabla\times\mathbf{A})_r = \dfrac{1}{r\sin\theta}\left(\dfrac{\partial(\sin\theta\, A_\varphi)}{\partial\theta} - \dfrac{\partial A_\theta}{\partial\varphi}\right)$
$(\nabla\times\mathbf{A})_y = \left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)$	$(\nabla\times\mathbf{A})_\varphi = \left(\dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r}\right)$	$(\nabla\times\mathbf{A})_\theta = \dfrac{1}{r\sin\theta}\dfrac{\partial A_r}{\partial\varphi} - \dfrac{1}{r}\dfrac{\partial(r A_\varphi)}{\partial r}$
$(\nabla\times\mathbf{A})_z = \left(\dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right)$	$(\nabla\times\mathbf{A})_z = \dfrac{1}{r}\left(\dfrac{\partial(r A_\varphi)}{\partial r} - \dfrac{\partial A_r}{\partial\varphi}\right)$	$(\nabla\times\mathbf{A})_\varphi = \dfrac{1}{r}\left(\dfrac{\partial(r A_\theta)}{\partial r} - \dfrac{\partial A_r}{\partial\theta}\right)$
	Laplacian of $\psi = \operatorname{div\,grad}\psi = \nabla^2\psi$	
$\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} + \dfrac{\partial^2\psi}{\partial z^2}$	$\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\psi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\psi}{\partial\varphi^2} + \dfrac{\partial^2\psi}{\partial z^2}$	$\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\psi}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial\psi}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2\psi}{\partial\varphi^2}$

Therefore

$$(\nabla \times \mathbf{A})_1 = \frac{1}{h_2 h_3} \left[\frac{\partial (h_3 A_3)}{\partial q_2} - \frac{\partial (h_2 A_2)}{\partial q_3} \right], \tag{7}$$

and the other two components are obtained by cyclic interchange of the coordinate indices.

In Table III–1 are listed the explicit forms for gradient, divergence, curl, and Laplacian in the three most common coordinate systems. For the definition of other othogonal coordinates see, e.g., Margenau and Murphy, *The Mathematics of Physics and Chemistry*.