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T II: Elektrodynamik

(Prof. E. Frey)

Problem set 8

Tutorial 8.1 *Helical waves*

Motivate the constitutive equations for the induced current density $\vec{j}(\vec{r},t)$ for the motion of electrons in a metal in the presence of an external time-independent and spatially uniform magnetic field $\vec{B}_{\text{ext}} = B_{\text{ext}}\hat{e}_z$ (Drude-Hall theory)

$$\partial_t^2 \vec{P}(\vec{r},t) + \frac{1}{\tau}\partial_t \vec{P}(\vec{r},t) + \frac{(-e)}{m^* c}\vec{B}_{\text{ext}} \times \partial_t \vec{P}(\vec{r},t) \quad = \quad \frac{\omega_p^2}{4\pi}\vec{E}(\vec{r},t)\,.$$

Here ω_p denotes the plasma frequency, m^* the effective mass of the electrons, and τ is a characteristic relaxation time.

a) Perform a temporal Fourier transform and show that in a spherical basis, $P_\pm = (P_x \mp \mathrm{i}P_y)/\sqrt{2}$, $P_0 = P_z$, and similarly for the electric field, the susceptibility tensor is diagonal,

$$P_\pm(\vec{r},\omega) = \chi_\pm(\omega)E_\pm(\vec{r},\omega)\,, \qquad P_0(\vec{r},\omega) = \chi_0(\omega)E_0(\vec{r})\,.$$

Determine the susceptibilities and infer the corresponding dielectric tensor. It is convenient to introduce the electron Larmor frequency $\omega_L = eB_{\text{ext}}/m^*c$.

b) For the remainder of the problem, neglect the damping of the electronic response, $\tau \to \infty$. Consider monochromatic plane waves with wave vector $\vec{k}$ parallel to the magnetic field $\vec{B}_{\text{ext}}$. Show that there are propagating circularly polarized, transverse waves and calculate their dispersion relation in the form $k = k_\pm(\omega)$ for high angular frequencies $\omega \gg \omega_p, \omega_L$.

c) Consider now a monochromatic electromagnetic wave of angular frequency $\omega \gg \omega_p, \omega_L$ incident on the metal that was originally linearly polarized in a direction transverse to the magnetic field. Show that after propagating a length L along the magnetic field, the plane of polarization is rotated (Faraday effect).

Tutorial 8.2 *Fermat's principle*

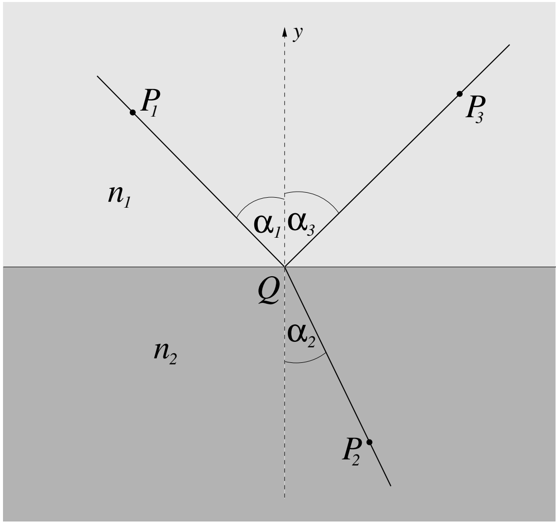
Fermat's principle states that the path taken between two points by a ray of light is the path that can be traversed in the least time, or more precisely, the optical path length must be extremal.

Consider an interface of two media with different indices of refraction, n_1 and n_2. A light ray from point P_1 in medium 1 hitting the interface at an angle of incidence α_1 is refracted to P_2 in medium 2 and reflected to P_3 in medium 1. The angles of refraction and reflection are α_2 and α_3, respectively.

Applying Fermat's principle, derive Snell's law of refraction,

$$n_1 \sin(\alpha_1) = n_2 \sin(\alpha_2),$$

and the law of reflection, $\alpha_1 = \alpha_3$.



Problem 8.3 *Optical activity*

A model for the optical rotation of a chiral material is given in terms of Condon's constitutive equations

$$\vec{D} = \varepsilon\vec{E} - g\partial_t\vec{H}\,, \qquad \vec{B} = \mu\vec{H} + g\partial_t\vec{E}\,,$$

where μ denotes the magnetic permeability, ε the average dielectric constant, and g abbreviates the *gyrotropic* constant. The characteristic length cg is on the atomic scale, hence for optical frequencies for which the model is built, $\omega g \ll \sqrt{\varepsilon\mu} = n$.

a) Show that for a monochromatic plane wave propagating along the z direction, Maxwell's equations imply

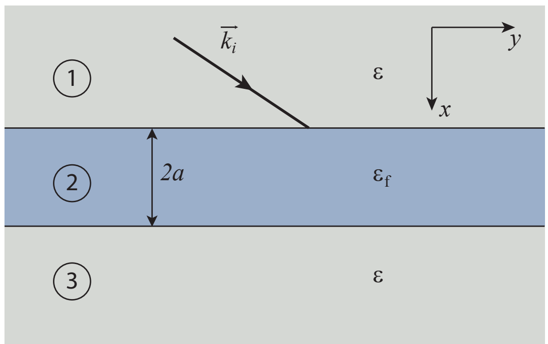
$$\left(k \mp g\frac{\omega^2}{c}\right) H_\pm = \mp\mathrm{i}\varepsilon\frac{\omega}{c}E_\pm\,, \qquad \left(k \mp g\frac{\omega^2}{c}\right) E_\pm = \pm\mathrm{i}\mu\frac{\omega}{c}H_\pm\,,$$

where $E_\pm = (E_x \mp \mathrm{i}E_y)/\sqrt{2}$ and similar for the magnetic field are evaluated in a spherical basis. Show that the waves in such a material are linear superpositions of two circularly polarized waves. Determine the corresponding dispersion relations in the form $k = k_\pm(\omega)$. Discuss also the phase relation between $D_\pm, E_\pm$ and $B_\pm, H_\pm$ to leading order in the small parameter g.

b) A linearly polarized plane wave is normally incident on such a medium. Discuss the propagation in the medium.

Problem 8.4 *Tunnel effect*

Consider a thin film of thickness $2a$ characterized by a dielectric constant ε_f dividing the three-dimensional space (with dielectric constant ε), see figure. A monochromatic electromagnetic plane wave is incident on the film; the coordinate system is chosen such that the incident wave vector reads $\vec{k}_\mathrm{i} = (k, k_\parallel, 0)$. Discuss the case of a polarization of the incident electric field parallel to the interfaces, $\vec{E}_\mathrm{i} = (0, 0, E_\mathrm{i})$, i.e., out of the drawing plane.



a) Determine the dispersion relation $\omega = \omega(\vec{k})$ separately in each region.

b) Argue that the polarization of the electric field is parallel to the interface in all three regions.

c) Since the tangential component of the electric field is continuous at the interfaces, the spatio-temporal modulations at the interface are identical. Justify the following Ansatz for the electric field

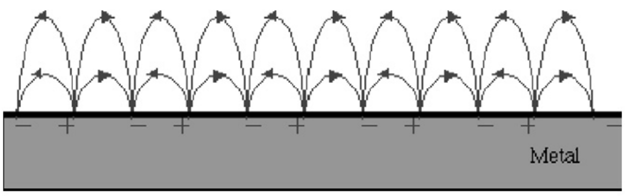
$$E_z(\vec{x}, t) = \mathrm{e}^{\mathrm{i}k_\parallel y - \mathrm{i}\omega t} \begin{cases} E_\mathrm{i}\mathrm{e}^{\mathrm{i}kx} + E_\mathrm{r}\mathrm{e}^{-\mathrm{i}kx} & \text{for } x < -a, \\ E_+\mathrm{e}^{\mathrm{i}qx} + E_-\mathrm{e}^{-\mathrm{i}qx} & \text{for } -a < x < a, \\ E_\mathrm{t}\mathrm{e}^{\mathrm{i}kx} & \text{for } x > a\,, \end{cases}$$

and interpret the individual terms. Show that q becomes purely imaginary for $k_\parallel^2 > \varepsilon_\mathrm{f}\,\omega^2/c^2$.

d) Establish the conditions of continuity for the tangential components of $\vec{H}$ $(= \vec{B}$ here) and calculate the effective transmission amplitude $t = E_\mathrm{t}/E_\mathrm{i}$ and the effective reflection amplitude $r = E_\mathrm{r}/E_\mathrm{i}$. Discuss the maxima of the transmission coefficient $T = |t|^2$ in the case of normal incidence. For total reflection, $k_\parallel^2 > \varepsilon_\mathrm{f}\,\omega^2/c^2$, interpret the asymptotic behavior of T for thick films.

Problem 8.5 *Surface-plasmon polaritons*

Consider an interface between a metal characterized by a dielectric function $\varepsilon_1(\omega) = 1 - \omega_p^2/\omega^2$ and an ideal dielectric, $\varepsilon_2(\omega) = \varepsilon = const$, $(\varepsilon > 1)$. In each material the constitutive equations $\vec{D}_i = \varepsilon_i \vec{E}_i, \vec{B}_i = \vec{H}_i, i = 1, 2$ apply. The interface supports electromagnetic modes propagating along the interface (surface-plasmon polaritons). Taking the interface as the $z = 0$ plane and choosing the propagation of the mode as the x-direction, choose as an Ansatz for the fields



Schematic representation of the electromagnetic field associated with a surface-plasmon polariton propagating along a metal-dielectric interface.

$$\vec{E}_i(\vec{r}, t) = \vec{\mathcal{E}}_i\, \mathrm{e}^{\mathrm{i}(qx - \omega t)}\mathrm{e}^{-\kappa_i |z|}$$
$$\vec{B}_i(\vec{r}, t) = \vec{\mathcal{B}}_i\, \mathrm{e}^{\mathrm{i}(qx - \omega t)}\mathrm{e}^{-\kappa_i |z|}$$

for $i = 1, 2$ and with positive decay constants $\kappa_i > 0$.

a) Formulate appropriate continuity conditions for the amplitudes $\vec{\mathcal{E}}_i, \vec{\mathcal{B}}_i$ across the interface, $z = 0$. Show that the magnetic fields are perpendicular to the interface and to the direction of propagation, i.e. $\vec{\mathcal{B}} = (0, \mathcal{B}_y, 0)$.

b) Sketch the dispersion $\omega = \omega(q)$. Show that for short wavelengths $q \gg \omega_p/c$, the surface-plasmon polariton frequency approaches a constant ω_s, whereas for long wavelength, the dispersion is linear $\omega = c_s q$ to first order in q. Discuss the attenuation length $l_i = 1/\kappa_i$ in the metal and the dielectric as a function of frequency.

Due date: Tuesday, 6/19/07, at 9 p.m.