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Prof. Dr. E. Frey
Dr. T. Franosch

Lehrstuhl für Statistische Physik
Biologische Physik & Weiche Materie
Arnold-Sommerfeld-Zentrum für Theoretische Physik
Department für Physik

Ludwig Maximilians Universität

T II: Elektrodynamik

(Prof. E. Frey)

Problem set 11

Tutorial 11.1 *Hertz potential*

a) Starting from Maxwell's equations introduce electromagnetic potentials and derive the wave equations

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{A} = \frac{4\pi}{c}\vec{j}\,, \qquad \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\varphi = 4\pi\rho$$

provided the potentials fulfill the Lorentz gauge $c^{-1}\partial_t\varphi + \operatorname{div}\vec{A} = 0$. Hence, the fields appear to be decoupled, but the gauge constraint has to be satisfied.

b) The continuity equation for the charge is solved automatically by introducing the polarization $\vec{P}$ with $\partial_t\vec{P} = \vec{j}$ and $-\operatorname{div}\vec{P} = \rho$. Show that the Lorentz gauge is satisfied automatically by introducing the (electric) Hertz potential $\vec{Z}$ with $c^{-1}\partial_t\vec{Z} = \vec{A}$ and $-\operatorname{div}\vec{Z} = \varphi$. What equation governs the dynamics of $\vec{Z}$? Express the electric and magnetic field in terms of the Hertz potential.

c) Perform a temporal Fourier transform and show that the solution compatible with Sommerfeld's radiation condition is given by

$$\vec{Z}_\omega(\vec{x}) = \int \mathrm{d}^3\vec{y}\,\frac{\vec{P}_\omega(\vec{y})}{|\vec{x}-\vec{y}|}\mathrm{e}^{\mathrm{i}k|\vec{x}-\vec{y}|}\,, \qquad k = \frac{\omega}{c}\,.$$

d) Argue that the leading behavior of the Hertz potential far away from the source is obtained by approximating

$$\vec{Z}_\omega(\vec{x}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\vec{g}(k\hat{n})\,, \qquad \vec{g}(\vec{k}) = \int \mathrm{d}^3\vec{y}\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{y}}\vec{P}_\omega(\vec{y})\,,$$

where $\vec{x} = r\hat{n}$, $|\hat{n}| = 1$ and determine the electric and magnetic field in the radiation zone.

e) In the case of a highly localized source, $kd \ll 1$, expand the exponential in $\vec{g}(\vec{k})$ in powers of $\vec{k}$ and show that the leading contributions to the angular dependence of the radiation are of the following form,

$$\vec{g}(\vec{k}) = \vec{p} - \hat{n}\times\vec{m} - \frac{1}{6}\mathrm{i}\vec{k}\cdot\underline{\underline{Q}}\ldots$$

Interpret the individual terms.

Problem 11.2 *Magnetic Hertz potential*

In addition to the electric Hertz potential $\vec{Z}$ one may also introduce a magnetic Hertz potential $\vec{Z}_m$.

a) Show that the Lorentz gauge is automatically fulfilled provided one chooses $\varphi = -\operatorname{div}\vec{Z}$ and $\vec{A} = c^{-1}\partial_t\vec{Z} + \operatorname{curl}\vec{Z}_m$. Show that the wave equations for the gauge potentials $\varphi, \vec{A}$ are fulfilled if one requires

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{Z} = 4\pi\vec{P} \quad \text{and} \quad \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{Z}_m = 4\pi\vec{M}.$$

Express the electromagnetic fields in terms of the two Hertz potentials.

b) Perform a temporal Fourier transform of the wave equation for the magnetic Hertz potential and give a formal solution that fulfills Sommerfeld's radiation condition. Show that the leading behavior of the magnetic Hertz potential in a long-wavelength expansion far away from the source is given by

$$Z_m^\omega(\vec{x}) = \frac{\mathrm{e}^{ikr}}{r}\vec{m}_\omega\,, \qquad r = |\vec{x}|\,, \quad k = \omega/c\,,$$

and interpret the vector $\vec{m}_\omega$.

c) Derive the corresponding electromagnetic fields $\vec{E}, \vec{B}$ as well as the time-averaged Poynting vector in the radiation zone. Discuss the angular dependence of the emitted radiation and the polarization of the electric field for the case of $\vec{m}_\omega = m_\omega(0,0,1)$.

Problem 11.3 *Gravitational waves*

As a relativistic invariant minimal generalization of Newton's theory of gravity consider the schematic model

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\Phi(\vec{x},t) = 4\pi G\rho(\vec{x},t)\,,$$

where $\Phi(\vec{x},t)$ denotes the gravitational potential, $\rho(\vec{x},t)$ the mass density, G the universal gravitational constant, and c the speed of light.

a) Perform a temporal Fourier transform, $\Phi_\omega(\vec{r}) = \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\,\Phi(\vec{r},t)$, and show that the (causal) solution for a localized source is given by

$$\Phi_\omega(\vec{x}) = G\int \mathrm{d}^3y\, \frac{\rho_\omega(\vec{y})}{|\vec{x}-\vec{y}|}\mathrm{e}^{\mathrm{i}k|\vec{x}-\vec{y}|}\,, \qquad k = \omega/c\,.$$

b) Argue that the leading behavior of the potential in the radiation zone $r \gg \lambda$ and far away from the localized source $r \gg d$ is obtained by approximating

$$\Phi_\omega(\vec{x}) \simeq G\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\int \mathrm{d}^3y\, \mathrm{e}^{-\mathrm{i}k\hat{n}\cdot\vec{y}}\rho_\omega(\vec{y}), \qquad \vec{x} = r\hat{n}, \quad |\hat{n}| = 1.$$

c) If the linear dimensions d of the radiating source are small compared to the wavelength, $kd \ll 1$, the exponential $\mathrm{e}^{-\mathrm{i}k\hat{n}\cdot\vec{y}}$ can be expanded in a power series. Determine the lowest order contribution in this long-wavelength expansion reflecting that the mass and momentum conservation laws hold

$$\partial_t\rho(\vec{x},t) + \nabla_i J_i(\vec{x},t) = 0\,, \qquad \partial_t J_i(\vec{x},t) + \nabla_j \Pi_{ij}(\vec{x},t) = 0\,,$$

where $\vec{J}(\vec{x},t)$ denotes the mass current also identified with the momentum density, and $\Pi_{ij}(\vec{x},t)$ corresponds to the momentum current.

Problem 11.4 *Rotating dipole*

Consider an electric dipole rotating in the x–y plane,

$$\vec{p}(t) = \mathrm{Re}\left(\vec{p}_\omega \mathrm{e}^{-\mathrm{i}\omega t}\right) , \qquad \vec{p}_\omega = p_\omega (1, \mathrm{i}, 0)/\sqrt{2}\,,$$

and discuss the emitted electromagnetic radiation.

a) Determine the angular dependence of the time-averaged Poynting vector for radiation emitted in the direction $\hat{n} = (\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)$. What is the total power radiated by the dipole?

b) Evaluate the polarization of the emitted radiation for directions $\hat{n}$
   (i) in the plane of rotation ($\vartheta = \pi/2$),
   (ii) perpendicular to it ($\vartheta = 0$).

Due date: Tuesday, 7/10/2007, at 9 p.m.