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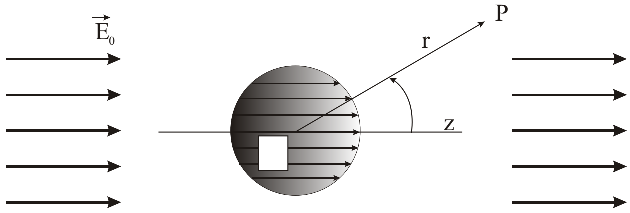
T II: Elektrodynamik

(Prof. E. Frey)

Problem set 10

Tutorial 10.1 *Dielectric sphere*

A dielectric sphere of radius R characterized by a dielectric constant ε is placed in an initially uniform electric field $\vec{E}_0$. For convenience, choose the center of the sphere as the origin and consider $\vec{E}_0$ along the z-axis. Then the problem exhibits an axial symmetry, which simplifies the problem.



a) Determine the electrostatic potential $\varphi(\vec{x})$ inside the sphere, $|\vec{x}| \le R$, and outside the sphere, $|\vec{x}| > R$. Formulate appropriate matching conditions at the surface of the sphere. Recall that the most general axially symmetric solution of Laplace's equation $\nabla^2\varphi = 0$ in polar coordinates, $\vec{x} = r(\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)$, is given in terms of

$$\varphi(\vec{x}) = \sum_{\ell=0}^{\infty}\left(a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right)\mathrm{P}_\ell(\cos\vartheta)\,,$$

where $\mathrm{P}_\ell(t)$ denotes the Legendre polynomials and a_ℓ, b_ℓ are undetermined coefficients.

b) Derive the corresponding electric field $\vec{E}(\vec{x})$. Extract the polarization $\vec{P}(\vec{x})$ inside the sphere and find the total induced dipole moment $\vec{p}$ of the dielectric sphere. Determine the effective polarizability of the sphere α defined by $\vec{p} = \alpha\vec{E}_0$.

c) Show that charges accumulate at the surface of the sphere and determine the induced surface charge density σ.

Problem 10.2 *Magnetic shielding – µ-metal*

A µ-metal is a nickel-iron alloy that has a very high magnetic permeability $\mu \sim 10^4 - 10^6 \gg 1$. The technical application of these materials is the screening of static (or low-frequency) magnetic fields, which cannot be attenuated by other methods.

Consider a spherical shell of magnetic permeability μ and inner an outer radii R_i and R_o, respectively, placed in a previously uniform magnetic field $\vec{H}_\infty$. The medium inside and outside of the shell has a magnetic permeability $\mu = 1$.

a) Argue that one can introduce a scalar magnetic potential φ_M to represent the field $\vec{H} = -\vec{\nabla}\varphi_M$, and show that it fulfills the Laplace equation $\nabla^2\varphi_M = 0$ in each region.

b) State the appropriate matching conditions for φ_M at the interfaces $r = R_\mathrm{i}$ and $r = R_\mathrm{o}$.

c) Recalling that the most general solution of the Laplace equation with cylindrical symmetry is provided by

$$\varphi_M(\vec{r}) = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right) \mathrm{P}_\ell(\cos\vartheta)\,,$$

determine the magnetostatic potential in each region. Calculate explicitly the corresponding magnetic field $\vec{H}$ inside of the shell.

d) Determine the leading behavior of the field inside of a thin shell of a µ-metal for $\mu \to \infty$. Discuss why a µ-metal provides an effective shielding.

Problem 10.3 *Legendre polynomials*

Consider the following partial differential equation

$$\frac{\partial}{\partial t}\left[(1-t^2)\frac{\partial\psi}{\partial t}\right] = -\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right). \qquad (*)$$

A solution is provided by the function

$$\psi(r,t) = \frac{1}{\sqrt{1-2rt+r^2}}\,, \qquad -1 < r < 1\,, \quad -1 \le t \le 1\,,$$

which serves as a generating function for the *Legendre polynomials* $\mathrm{P}_\ell(t)$, i.e., a Taylor expansion with respect to r, $\psi(r,t) = \sum_{\ell=0}^{\infty} r^\ell \mathrm{P}_\ell(t)$, defines the functions $\mathrm{P}_\ell(t)$.

a) Show by explicit substitution that $\psi(r,t)$ indeed solves the partial differential equation $(*)$.

b) Identifying $t = \cos\vartheta$ reveals that $\psi(r,t)$ corresponds to the Coulomb potential of a unit charge located on the z-axis at unit distance from the origin. Thus $\psi(r, t = \cos\vartheta)$ solves the Laplace equation in polar coordinates

$$\nabla^2\psi(r,\cos\vartheta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\psi}{\partial\vartheta}\right) = 0\,.$$

Using this observation derive the partial differential equation $(*)$.

c) Determine explicitly $\mathrm{P}_\ell(t = \pm 1)$ observing that for $t = \pm 1$ the Taylor series of $\psi(r,t)$ becomes elementary.

d) Employ the symmetries of $\psi(r,t)$ to argue that $\mathrm{P}_\ell(t)$ is a symmetric (anti-symmetric) function for even (odd) ℓ.

e) Inspect the Taylor series to demonstrate that $\mathrm{P}_\ell(t)$ is a polynomial of order ℓ.
Hint: Expand the square root in $x = 2rt - r^2$.

f) Calculate and sketch the first four Legendre polynomials ($\ell = 0, \ldots 3$).

g) Substitute the Taylor series of $\psi(r,t)$ in the partial differential equation $(*)$. Comparing the coefficients of r^ℓ confirm that the $\mathrm{P}_\ell(t)$ satisfy the second order differential equation, i.e., they are indeed Legendre polynomials,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t^2)\frac{\mathrm{dP}_\ell(t)}{\mathrm{d}t}\right] + \ell(\ell+1)\mathrm{P}_\ell(t) = 0\,, \qquad -1 \le t \le 1\,. \qquad (**)$$

h) Show that $(1-2rt+r^2)\partial\psi/\partial r = (t-r)\psi$. Make use of this result to derive the recursion relation

$$\ell\,\mathrm{P}_{\ell-1}(t) - (2\ell+1)t\,\mathrm{P}_\ell(t) + (\ell+1)\,\mathrm{P}_{\ell+1}(t) = 0\,.$$

Similarly, verify that $(1-2rt+r^2)\partial\psi/\partial t = r\psi$ and prove

$$\mathrm{P}'_{\ell+1}(t) - 2t\mathrm{P}'_\ell(t) + \mathrm{P}'_{\ell-1}(t) = \mathrm{P}_\ell(t)\,.$$

i*) Show that the Legendre polynomials are orthogonal in the following sense,

$$\int_{-1}^{1} \mathrm{d}t\,\mathrm{P}_\ell(t)\mathrm{P}_{\ell'}(t) = \frac{2}{2\ell+1}\delta_{\ell\ell'}\,.$$

Hint: Employ the differential equation $(**)$ to show that

$$[\ell'(\ell'+1) - \ell(\ell+1)]\int_{-1}^{1} \mathrm{d}t\,\mathrm{P}_\ell(t)\mathrm{P}_{\ell'}(t) = 0$$

and conclude that orthogonality holds. The normalization follows by considering $\int \mathrm{d}t\,\psi(r,t)^2$. First perform the integration directly; then use the Taylor expansion in r and the orthogonality property.

Due date: Tuesday, 7/3/07, at 9 p.m.