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SoSe 05

Set 1
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T IV: Thermodynamik und Statistik
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Problem set 1

Problem 1.1 *Maxwell distribution*
The probability density for a particle in a fluid to have a velocity $\mathbf{v} = (v_x, v_y, v_z)$ is

$$p(\mathbf{v}) = \mathcal{N} \exp \frac{-M}{2k_B T} \mathbf{v}^2 \, ,$$

where M, k_B, T are some positive constants. Evaluate the missing normalization factor $\mathcal{N}$. Find the average $\langle v \rangle$ of the velocity $v = |\mathbf{v}|$ and the average kinetic energy $\langle E \rangle = M \langle v^2 \rangle / 2$. Compare the kinetic energy of a particle that moves with the mean velocity to the mean kinetic energy.

Problem 1.2
ϕ is a random phase angle distributed uniformly over the range 0 to 2π and

$$x = \cos\phi \, , \qquad y = \sin\phi$$

Calculate the probability distribution of x and y and the joint probability distribution of x and y. Evaluate the covariance $\langle (x - \langle x \rangle)(y - \langle y \rangle) \rangle$. Are the variables x and y statistically independent?

Problem 1.3 *Noninteracting spins*
A system with m spins without any external field or interaction between the spins has equal probability for a single spin to be up or down.
(a) Write down the probability for having n spins up and $m - n$ down.
(b) Show $\sum_{n=0}^{m} w(m, n) = 1$.
(c) Calculate the mean $\langle n \rangle$ and the variance $\langle \Delta n^2 \rangle^{1/2}$ of n.
(d) The dimensionless magnitization is defined by $M = 2n - m$. Calculate its mean and variance.
(e) Calculate the distribution $w(m, n)$ for small deviations x from the mean value $\langle n \rangle$ and large m, i.e. $|x| \ll \langle n \rangle$.

Problem 1.4 *Characteristic Functions*
For a probability density $p(x)$ the corresponding characterstic function is defined as

$$C(\xi) \equiv \langle e^{i\xi x} \rangle = \int e^{i\xi x} p(x) dx \, .$$

Demonstrate the following properties:
(a) $C(0) = 1$.
(b) $|C(\xi)| \leq C(0)$.
(c) $C(\xi)$ is continuous on the real axis, even if $p(x)$ has discontinuities.
(d) $C(-\xi) = C(\xi)^*$
(e) $C(\xi)$ is positive semi-definite, i.e. for an arbitrary set of N real numbers $\xi_1, \xi_2, .., \xi_N$ and N arbitrary complex numbers $a_1, a_2, .., a_N$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i^* a_j C(\xi_i - \xi_j) \geq 0 \, .$$

Problem 1.5 *Moment generating function and cumulants*

For some probability densities $p(x)$ the moment generating function

$$M(\xi) \equiv \langle e^{x\xi} \rangle = \int e^{x\xi} p(x) dx$$

is well-defined for real ξ. Expand $M(\xi)$ in powers of ξ, $M(\xi) = \sum_{r=0}^{\infty} \nu_r \xi^r / r!$ and relate the numbers ν_r to the moments of $p(x)$. Another useful function is $K(\xi) = \ln M(\xi)$ known as the *cumulant generating function.* The power expansion with respect to ξ reads $K(\xi) = \sum_{r=1}^{\infty} \kappa_r \xi^r / r!$ with coefficents κ_r referred to as *cumulants.*

(a) Relate the first five cumulants $\kappa_1, .., \kappa_5$ to the numbers $\nu_1, .., \nu_5$.

(b) Evaluate $M(\xi)$, the first three moments and cumulants for

(i) the *Bernouilli* distribution

$$p_n = \binom{N}{n} \beta^n (1-\beta)^{N-n} , \qquad 0 \le n \le N \, , 0 \le \beta \le 1 \, .$$

(ii) the *Poisson* distribution

$$p_n = \frac{\lambda^n}{n!} e^{-\lambda} , \qquad \lambda > 0 \, , \quad n = 0, 1, \ldots$$

(iii) the *Bose-Einstein* distribution

$$p_n = (1-\eta)\eta^n , \qquad 0 \le \eta < 1 \, , \quad n = 0, 1, \ldots$$

(iv) the *Gaussian* distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2} \right) , \qquad \sigma > 0 \, .$$

Problem 1.6 *Master equation*

The dynamics of some system of N states satisfies the master equation

$$\frac{d}{dt} w_i(t) = \sum_{k=1}^{N} \Pi_{ik} w_k(t) \, .$$

Here $w_i(t), i = 1, .., N$ denote the probabilities to find the state i at time t. The transition matrix Π_{ik} reads

$$\Pi_{ik} = \mu - \mu N \delta_{ik} \, , \qquad \mu > 0$$

(a) Demonstrate the conservation of probability, i.e. $\sum_{i=1}^{N} w_i(t) = 1$ for all times, provided that $\sum_{i=1}^{N} w_i(t = 0) = 1$.
(b) Show the existence of an *equilibrium distribution*, i.e. a stationary distribution.
(c) Verify the formal solution $\underline{w}(t) = \exp(\underline{\underline{\Pi}} t) \underline{w}(t = 0)$ in obvious vector notation.
(d) Find all eigenvalues and eigenvectors of $\underline{\underline{\Pi}}$ and calculate the complete solution of the master equation.

Problem 1.7 *Umbrella problem*

On a rainy April day a group of n students all equipped with umbrellas walk at noon to the TU Mensa. They leave their umbrellas in the hall during lunch. Returning in a hurry everybody picks randomly one of the previously deposited umbrellas. Calculate the probability that *nobody* returns with his/her own umbrella. If you cannot find a general formula evaluate the probabilities up to $n = 8$. What do you anticipate for large groups?